An Abduction-based Method for Explaining Non-entailments of Semantic Matching

Ivan Gocev [a,*], Georgios Meditskos [a] and Nick Bassiliades [a]
[a] *School of Informatics, Aristotle University of Thessaloniki, Greece*
E-mails: ivangochev@csd.auth.gr, gmeditsk@csd.auth.gr, nbassili@csd.auth.gr

Abstract. Explainable Artificial Intelligence (XAI) attempts to give explanations for decisions made by AI systems. Despite the fact that for knowledge-based systems this is perceived as inherently easier than for black-box AI systems based on Machine Learning, research is still required for computing satisfactory explanations of reasoning results. In this paper, we focus on explaining non-entailments as a result of semantic matching in $\mathcal{EL}_\perp$ ontologies. In the cases where the result of semantic matching is an entailment, the already established methods of justifications and proofs provide excellent results. On the other hand, the cases in which the result of semantic matching presents in the form of a non-entailment demand an alternative approach. Inspired by abductive reasoning techniques, we present a method for computing subtree isomorphisms between graphical representations of $\mathcal{EL}_\perp$ concept descriptions, which are then used to construct solutions to abduction problems, i.e. explanations, for semantic matching non-entailments in $\mathcal{EL}_\perp$ ontologies. We then illustrate our method with an example scenario and discuss the results.

Keywords: Explanations, Description Logics, Semantic Matching, Non-entailments, Abduction

1. Introduction

Today, the vast amount of data that is available has driven research into Machine Learning (ML) and Deep Learning approaches, which provide AI systems with the possibility to autonomously learn and adapt to various scenarios. These models consist of black-box structures, i.e. they provide outcomes without revealing any information about their underlying workings. To this end, with knowledge based systems being inherently explainable, explanations techniques are more focused on black-box systems [1]. However, the information that these approaches provide is often numerical, which lacks context and needs additional information to understand how a result was achieved or to interpret that result. Knowledge based systems and reasoning are well suited to provide help here, by encoding information and adding context to it, as well as providing the possibility to reason with that context. In addition, explanation techniques in symbolic AI could help, when integrated with sub-symbolic AI, to understand the underlying workings of sub-symbolic approaches. Furthermore, the existing research in explainability for knowledge base systems is not complete and has room for improvement. E.g., the already established methods of justifications [10] and proofs [14] provide excellent results when explaining logical inferences of knowledge bases. However, they fall short when explaining why observations are not a logical consequence of some knowledge base [15, 16, 38], when, in reality, they should be. Widely used approaches that are based on reasoning techniques in Symbolic AI, for example semantic matching, are enormously enhanced when explanations are introduced. Thus, research of explanation techniques for knowledge-based systems, even though perceived inherently easier than for black-box AI systems, is still a vibrant topic.

*Corresponding author. E-mail: ivangochev@csd.auth.gr.

Ontologies offer a suitable way to capture and classify domain knowledge from human experts, making them practical in various substantial fields. SNOMED CT[1] is a standardized medical ontology that provides codes and definitions for medical terms [2], and the ChEBI ontology[2] represents chemical entities of biological interest [3], with both including over 300,000 axioms. In addition, ontologies have been developed for various industrial fields such as electronics [4], energy [5], process engineering [6], and construction [7], to name a few. Besides modeling domain knowledge, ontologies offer numerous ways to support users' decision making [8, 9]. This is achieved by means of logical reasoning, which provides the possibility to perform reasoning tasks such as consistency checking, instance retrieval, and inference. To further enhance the decision-making support, methods that provide explanations for reasoning results are included in knowledge-based systems [10, 14], thus creating explainability within the scope of Symbolic AI.

Within [18], semantic matchmaking (or semantic matching) and its most widely used degrees of matches are introduced, which are: exact, plugin, subsume and intersection. In the context of intersection based matching, [20] offers experiences and modeling guidelines to achieve a robust matchmaking service. In [21], the degrees of matches are applied to semantic descriptions that capture the service "as a whole", represented by a concept in a knowledge base. In [22], an approach for matching service descriptions based on WSMO framework is presented and in [23] a web service matchmaking algorithm is presented that can be used to process an initial set of candidate services in order to aid the search of more fine-grained discovery approaches. On the other hand, [24] present the notion of stateless services and an approach for the matching of services with high precision and recall, as well as prove that matching can be reduced to conjunctive query equivalence w.r.t. Tbox. This shows the wide use of semantic matching in Semantic Web Technologies. Thus, generating explanations for semantic matching could enhance scenarios that utilize this vastly used approach.

Explaining semantic matching helps make AI systems transparent and understandable. For example, in [42] semantic matching is used in an industrial scenario. At first ontologies are used to represent machine knowledge and semantic matching is used to match machines' capabilities with certain product requirements. Then, explanations for the outcomes of semantic matches are generated, which help users understand the reasons why a certain product cannot be produced. [46] presents a satisfiability-based approach to explain results of semantic matching and in [19, 47] we can see the utilization of semantic matching for an e-marketplace. To discover potential matches, they use abduction, which in a way explains why a negative result of a semantic match is obtained, by showing how the negative result can be converted into a positive result. In addition, semantic matching is widely known to be used in service discovery [22, 52].

Generating explanations for results of semantic matching in Description Logics (DLs) still poses interesting questions. The idea is to use semantic matching to identify whether concepts are, or are not, semantically related w.r.t. some background knowledge and explain that result. There are various formulations of the problem that have previously been proposed [46, 47, 49, 50, 54]. In general, the idea is to find meaningful reasons that explain the results of semantic matching, while retaining the original knowledge as much as possible. However, in some cases, the results of semantic matching may present in forms that complicate the formulation of explanations. This paper addresses those cases of semantic matching.

Our main contribution in this paper represents a method for explaining non-entailments as a result of semantic matching for the description logic $\mathcal{EL}_{\bot}$. In general, the problem of generating explanations for non-entailments is brought down to a search for relations between concepts and/or roles/role restrictions. To obtain these relations, most methods impose constraints on the set of concepts and/or role restrictions that can be included in explanations. These constraints usually allow for abduction of concepts only, and omit role restrictions. In addition, they limit the types of complex expressions that can be included in explanations. The challenge here is to minimize those constraints and allow for all forms of complex expressions to be included in explanations. To this end, we present our approach using subtree isomorphisms and illustrate how abductive solutions, that contain both concepts and role restrictions, can be constructed. We present a semantic model for the representation of complex definitions consisting of concepts and roles, which is constrained only by the signature of knowledge bases, and formalize our

[1] https://www.snomed.org/
[2] https://www.ebi.ac.uk/chebi/

approach. Finally, we present an implementation of our method and illustrate the results for a working example, as well as present experimental results from a synthetically constructed benchmark.

The remaining of the paper is structured as follows: Section 2 presents related work on abduction and methods for generating explanations of non-entailments in DLs, how explanation techniques are used for semantic matching, and limitations and challenges in the areas, Section 3 overviews the Description Logic $\mathcal{EL}_\bot$ and semantic matching in DLs, as well as related notions, in Section 4 we define the problem of explaining non-entailments obtained as a result of semantic matching, in Section 5 we present our methodology, Section 6 contains the implementation of our method presented on a working example, as well as results obtained from an experimental setup, Section 7 contains a brief comparison of our method to existing ones and finally, Section 8 contains concluding remarks.

2. Related Work

The problem of explaining results of semantic matching has been treated in various different ways in the past. They can be outlined by two main categories: explaining entailments, and explaining non-entailments. An entailment is an axiom that logically follows from a knowledge-base, e.g. an ontology or a knowledge graph. They are often referred to as logical inferences. One established method for explaining entailments are justifications [10]. A justification is a subset consisting of the minimal number of axioms from an ontology such that the entailment holds. Recently, proofs, in the form of directed hypergraphs, have been utilized to explain inferences for a defined logic $\mathcal{L}$, which could be either first-order logic or some description logic [11–14]. These methods provide excellent results when explaining entailments and, when a result of semantic matching presents as such, they can be applied directly. However, these methods fall short when explaining non-entailments, i.e. axioms that do not logically follow from a knowledge base. In this case, a standard approach is abduction [27, 29, 38, 41, 47, 49, 50, 54].

An abduction problem consists of a background knowledge and an observation that is not entailed. A solution, referred to as a hypothesis, is a set of axioms that when added to the background knowledge the observation becomes entailed. Since the hypothesis provides reasons for why the observation was not entailed in the first place, it is treated as an explanation for the non-entailment. Depending on the background knowledge and the observation, we can perform concept abduction [38, 39], where the hypothesis consists of atomic concepts restricted by the background knowledge, ABox abduction [27, 29–37], where the background knowledge includes assertions and the observation is a query to the ABox (the hypothesis contains assertive knowledge), TBox abduction [15–17], specific for explaining observations in the form of general concept inclusion axioms, and knowledge base abduction [28, 29], where the hypothesis consists of terminological and assertive axioms constructed from the background knowledge.

Naturally, in abduction, a set of possible abducibles is determined [16, 29]. Such a set provides the possible concepts or axioms that could be abducted and it represents a form of a constraint on the hypothesis. Recently, homomorphisms have been used to generate abductive solutions for TBoxes [15], which do not explicitly constraint the abduction to a set of predetermined abducibles. Still, they capture entailments through axioms that contain atomic concepts only. Though they extend the set of common minimality criteria [27], such as subset, size, and semantic minimality, and filter hypotheses that do not carry meaningful information, homomorphisms restrict hypotheses to concepts only and exclude role restrictions in abductive solutions, thus dismissing other potential abductive solutions. For example, considering TBox abduction, abductive solutions could be restricted to only include axioms of a certain type, such as concept inclusions of the form $A \sqsubseteq B$, containing only atomic concepts. However, in reality, the abductive solution may not only consist of such axioms. The key reason why an observation is not entailed may be a role restriction on a certain concept, represented by an axiom consisting of role restrictions as well, such as $A \sqsubseteq \exists r.B$. If we omit role restrictions in hypotheses, then an abductive solution may not be found even if it does, in fact, exist. This challenge is emphasized in [16]. To resolve it, they generate predefined patterns based on justifications, which represent forms of axioms that can be abducted.

We can also see the utilization of abduction for semantic matching in [45, 47, 54], where a search is propagated for the assumptions needed for a supply to meet a demand. The focus is mainly on the abduction of concepts that are added or contracted from definitions of matching concepts, in order to achieve a certain degree of semantic similarity between them. The added concepts represent hypotheses that explain the negative outcome of the semantic match. However, this contraction or extension may change the original concept definitions and the explanations may not

present as meaningful. To tackle this, specific constraints are introduced in the search for explanations. Our interest lies in preserving the definitions of matching concepts by finding direct relations between them, while putting minimal constraints on the process of abduction.

The problem of evaluating concept compatibility in semantic matching can be brought down to a search for a conjunction (or an extension of a conjunction) for given matching concept definitions. Ultimately, this search can be transformed into the subgraph isomorphism problem [51, 52]. We present how the subgraph isomorphism problem can be utilized to explain non-entailments of semantic matching and a method that computes subtree isomorphisms to generate explanations for non-entailments of semantic matching.

Similarly to [45, 47, 54], abduction is used to generate explanations. However, our method focuses on obtaining direct relations between matching concepts, i.e. general concept inclusions (GCIs), instead of concepts alone, to explain non-entailments of semantic matching. This approach preserves the structure of the definitions of matching concepts, thus filtering explanations that do not provide critical information for the outcome of semantic matching.

While graphs and homomorphisms have been used previously to solve abduction problems and to explain general concept inclusion non-entailments [15], our method generates explanations of non-entailments obtained as a result of semantic matching. In addition, we use subtree isomorphisms instead of homomorphisms between graphical representations of concept definitions to allow for abduction of not only concepts, but also role restrictions, which represents our major contribution in this paper.

Previously, in our work [41] we presented an approach for generating explanations for non-entailments in DLs using subtree isomorphisms. The approach is focused on explaining general concept inclusion non-entailments. Since the method in [41] finds direct relations between concepts in a non-entailment, it fits the use-case of semantic matching. In this paper we formalize our previous work for semantic matching and show how it can be used to generate explanations for results of semantic matching that are not entailed. We extend the definitions for subtrees and show in detail how subtree isomorphisms capture concept inclusions in semantic matching. In addition, we fully implement and test our method.

3. Preliminaries

In this section, we briefly summarize the description logic $\mathcal{EL}_\bot$, which is an extension of the description logic $\mathcal{EL}$ that also allows concept disjointness, along with other fundamental concepts and notations used throughout this paper.

3.1. $\mathcal{EL}_\bot$ Description Logic

We start by defining countably infinite disjoint sets, $\mathcal{N}_\mathcal{C}$ whose members are called *atomic concepts*, and $\mathcal{N}_\mathcal{R}$ whose members are called *roles*. By combining these we form a *signature*, $\Sigma := \langle \mathcal{N}_\mathcal{C}, \mathcal{N}_\mathcal{R} \rangle$. We denote atomic concepts by A_i, roles by r_i, and complex concepts C_i and/or D_i for $i \in \mathbb{N}^0$. In the case where a distinct concept and/or role occurs, the indexing is omitted and we write A for atomic concept, r for role, and C and/or D for a complex concept.

$\mathcal{EL}_\bot$ *concepts* (a.k.a. *complex concepts*) are inductively defined by the syntax:

$$C, D := \top \mid \bot \mid A \mid C \sqcap D \mid \exists r.C,$$

where $A \in \mathcal{N}_\mathcal{C}$, $r \in \mathcal{N}_\mathcal{R}$.

For concise representation, we denote a conjunction of $\mathcal{EL}_\bot$ concepts by $C_0 \sqcap C_1 \sqcap \cdots \sqcap C_n = \prod_{i=0}^{n} C_i$.

The *interpretation* $\mathcal{I} = (\Delta^\mathcal{I}, \cdot^\mathcal{I})$ consists of a non-empty set called the *domain* of $\mathcal{I}$ and a function $\cdot^\mathcal{I}$. The function $\cdot^\mathcal{I}$, defined on the sets of atomic concepts $\mathcal{N}_\mathcal{C}$ and roles $\mathcal{N}_\mathcal{R}$, associates with each concept name $A \in \mathcal{N}_\mathcal{C}, A^\mathcal{I} \subseteq \Delta^\mathcal{I}$, and with each role name $r \in \mathcal{N}_\mathcal{R}, r^\mathcal{I} \subseteq \Delta^\mathcal{I} \times \Delta^\mathcal{I}$. We can extend the *interpretation function* $\cdot^\mathcal{I}$ to complex concepts, by inductively defining:

- $\top^\mathcal{I} := \Delta^\mathcal{I}$, $\bot^\mathcal{I} := \emptyset$,

- $(C \sqcap D)^{\mathcal{I}} := C^{\mathcal{I}} \cap D^{\mathcal{I}}$,
- $(\exists r.C)^{\mathcal{I}} := \{a \in \Delta^{\mathcal{I}} \mid \exists b \in \Delta^{\mathcal{I}}, (a,b) \in r^{\mathcal{I}} \wedge b \in C^{\mathcal{I}}\}$.

A *TBox* $\mathcal{T}$ represents terminological knowledge and is a finite set of General Concept Inclusions (GCIs, a.k.a. axioms) of the forms $C \sqsubseteq D$ - we say "*C is subsumed by D*" with the meaning "*C is included in D*" or "*D includes C*" and $C \sqsupseteq D$ - we say "*C subsumes D*" with the meaning "*D is included in C*" or "*C includes D*".

We denote axioms by α and add indexing when there are multiple axioms, α_i for $i \in \mathbb{N}^0$. The interpretation $\mathcal{I}$ is a model of $C \sqsubseteq D$ if $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$ and it is written as $\mathcal{I} \models C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$. Similarly, $\mathcal{I}$ is a model of $C \sqsupseteq D$ if $C^{\mathcal{I}} \supseteq D^{\mathcal{I}}$, written as $\mathcal{I} \models C^{\mathcal{I}} \supseteq D^{\mathcal{I}}$. $\mathcal{I}$ is a model of $C \equiv D$ if $\mathcal{I}$ is a model of both $C \sqsubseteq D$ and $C \sqsupseteq D$. If the interpretation $\mathcal{I}$ is a model of every axiom in a TBox $\mathcal{T}$, then it is a model of $\mathcal{T}$. A TBox $\mathcal{T}$ is consistent if it has a model.

Definition 1. *A TBox $\mathcal{T}$ entails an axiom α if and only if all models of $\mathcal{T}$ satisfy α. In this case, we write $\mathcal{T} \models \alpha$.*

Definition 2. *A TBox $\mathcal{T}$ does not entail an axiom α if there exists a model of $\mathcal{T}$ that does not satisfy α. In this case, we write $\mathcal{T} \not\models \alpha$.*

When an axiom α is entailed by a TBox $\mathcal{T}$, $\mathcal{T} \models \alpha$, we refer to it as an *entailment* and say that α is entailed or α is a logical inference (consequence) of $\mathcal{T}$. When α is not entailed by $\mathcal{T}$, $\mathcal{T} \not\models \alpha$, we call it a *non-entailment* and say that α is not entailed or α is not a logical inference (consequence) of $\mathcal{T}$. The subsumption problem for the $\mathcal{EL}$ family of description logics is decidable in *polynomial time* [43].

3.2. Rooted Trees

We introduce some fundamental concepts and notations from graph theory, that are used throughout this paper.
A directed rooted tree is a tuple $T = \langle \mathcal{V}, \mathcal{E}, \rho \rangle$, where:

- $\mathcal{V}$ is a set of nodes,
- $\mathcal{E}$ is a set of edges, which are ordered pairs of distinct nodes $\mathcal{E} = \{\langle x, y \rangle \mid \langle x, y \rangle \in \mathcal{V}^2 \text{ and } x \neq y\}$,
- ρ is the root node.

We denote nodes of trees by $v_i, w_i, u_i, x_i, y_i, z_i$ for $i \in \mathbb{N}^0$ and edges of trees e_k for $k \in \mathbb{N}$. When referring to a distinct node or edge we omit the indexing and write v, w, u, x, y, z for nodes, and e for edges. We use the notation $\langle v_i, v_j \rangle$ to represent an edge e_k. In the edge $e_k = \langle v_i, v_j \rangle$, directed from v_i to v_j, the nodes v_i and v_j are called the *endpoints* of the edge, v_i is the *head* of the edge and v_j is the *tail* of the edge. Edges with same tail nodes are not allowed.

Let $T = \langle \mathcal{V}, \mathcal{E}, v_0 \rangle$ be a rooted tree. A path π in T is a sequence of nodes $v_1, v_2, \ldots, v_i, \ldots, v_n$ in $\mathcal{V}$ and a sequence of edges $e_1, e_2, \ldots, e_k, \ldots, e_{n-1}$ in $\mathcal{E}$, such that π begins with v_1 and ends with v_n and for each subsequent edges e_k and e_{k+1}, the tail node of e_k is the head node of e_{k+1}. All nodes and all edges in π are distinct. We denote by $\pi(v_1, v_n) = \langle v_1, \ldots, v_n \rangle$ for $v_1, \ldots, v_n \in \mathcal{V}$ the sequence of nodes for the path starting in v_1 and ending in v_n, and denote by $\pi_e(v_1, v_n) = \langle e_1, \ldots, e_{n-1} \rangle$ for $e_1, \ldots, e_{n-1} \in \mathcal{E}$ the sequence of edges for the path starting in v_1 and ending in v_n. $\pi(v_1, v_n)$ and $\pi_e(v_1, v_n)$ are different representations that refer to the same path. In a rooted tree there is a unique path between any two nodes. The length of a path is the number of vertices in a path, and is denoted by $|\pi(v_1, v_n)|$. If v_i lies on the distinct path from the root to v_n, then v_i is called an an *ancestor* of v_n and v_n is a *descendant* of v_i.

The *depth*, $d(v_i)$, of a node v_i is the number of edges in the path $\pi_e(v_0, v_i)$ and is $d(v_i) = |\pi_e(v_0, v_i)| = |\pi(v_0, v_i)| - 1$. The *children* of a node v_i are defined as $N(v_i) := \{v_j \mid v_j \in \mathcal{V} \text{ and } \langle v_i, v_j \rangle \in \mathcal{E}\}$. The *degree* of a node $v_i \in \mathcal{V}$ is $\delta(v_i) := |N(v_i)|$. We refer to a path as *complete* if it starts at the root node and ends in a leaf node. The set of all complete paths Π^0 in T is $\Pi^0 := \{\pi(v_0, v_i) \mid v_0, v_i \in \mathcal{V}, i \neq 0 \text{ and } \delta(v_i) = 0\}$. When we say $\pi(v_0, v_i) \in \Pi^0$ we also imply $\pi_e(v_0, v_i) \in \Pi^0$ and vice versa.

Nodes and edges of rooted trees can contain labels. A labeling is simply a map $f : A \mapsto B$, such that for every element $b \in B$ there is at least one element $a \in A$, such that $b = f(a)$. We can define a node-labeled rooted tree as follows:

Definition 3. *A rooted tree $T = \langle \mathcal{V}, \mathcal{E}, v_0, \lambda_{\mathcal{V}} \rangle$ is called node-labeled if there exists a labeling $\lambda_{\mathcal{V}} : \mathcal{V} \mapsto L_{\mathcal{V}}$, which maps all nodes to labels in a set of node labels $L_{\mathcal{V}}$, such that for any $v_i \in \mathcal{V}$ and $l_{\mathcal{V}} \in L_{\mathcal{V}}$, if v_i is mapped to $l_{\mathcal{V}}$, then $\lambda_{\mathcal{V}}(v_i) = l_{\mathcal{V}}$.*

Similarly, we can define an edge-labeled rooted tree.

Definition 4. *A rooted tree $T = \langle \mathcal{V}, \mathcal{E}, v_0, \lambda_{\mathcal{E}} \rangle$ is called edge-labeled if there exists a labeling $\lambda_{\mathcal{E}} : \mathcal{E} \mapsto L_{\mathcal{E}}$, which maps all edges to labels in a set of edge labels $L_{\mathcal{E}}$, such that for any $e_k \in \mathcal{E}$ and $l_{\mathcal{E}} \in L_{\mathcal{E}}$, if e_k is mapped to $l_{\mathcal{E}}$, then $\lambda_{\mathcal{E}}(e_k) = l_{\mathcal{E}}$.*

3.3. Semantic Matching

Semantic matching is a technique used to identify semantically related concepts [18, 19, 25, 26, 44]. Introduced in [18], semantic matching determines whether two concepts are semantically similar (or compatible) w.r.t. some background knowledge, if their intersection is satisfiable. If the intersection of two concepts is not satisfiable w.r.t. some background knowledge, then they are not semantically similar, i.e. the concepts are incompatible. We refer to the former as *positive intersection matches* and the latter as *negative intersection matches*.

Definition 5. *Let $\mathcal{T}$ be an $\mathcal{EL}_\bot$ TBox and C and D concepts defined w.r.t. $\Sigma_{\mathcal{T}} = \langle N_{\mathcal{C}}, N_{\mathcal{R}} \rangle$. Positive and negative intersection matches, respectively, occur when:*

$$\mathcal{T} \not\models C \sqcap D \equiv \bot, \tag{1}$$

$$\mathcal{T} \models C \sqcap D \equiv \bot. \tag{2}$$

A positive result of an intersection match (or a positive intersection match) occurs if two concepts C and D are semantically similar. Consequently, a negative result of an intersection match (or a negative intersection match), occurs if two concepts C and D are not semantically similar. Since it is not particularly useful to merely determine if two concepts are semantically similar or not, intersection matches are extended to: *exact*, *plugin*, and *subsume*, as seen in [18, 44]. It would be more useful to know whether they are, or could be, in a higher matching degree. The exact, plugin, and subsume match levels tell us the extent to which the concepts are semantically similar, which is more useful in decision-making scenarios.

Positive and negative semantic matches of higher degrees are defined as follows:

Definition 6. *Let $\mathcal{T}$ be an $\mathcal{EL}_\bot$ TBox and C and D concepts defined w.r.t. $\Sigma_{\mathcal{T}} = \langle N_{\mathcal{C}}, N_{\mathcal{R}} \rangle$. Positive semantic matches of higher degrees occur when:*

$$\mathcal{T} \models C \equiv D, \tag{3}$$

$$\mathcal{T} \models C \sqsubseteq D, \tag{4}$$

$$\mathcal{T} \models C \sqsupseteq D. \tag{5}$$

and negative semantic matches of higher degrees occur when:

$$\mathcal{T} \not\models C \equiv D, \tag{6}$$

$$\mathcal{T} \not\models C \sqsubseteq D, \tag{7}$$

$$\mathcal{T} \not\models C \sqsupseteq D. \tag{8}$$

Let C and D be two concepts defined w.r.t. the signature of some background knowledge. We denote by $\text{match}_{\sqcap}(C, D)$ the intersection match between the concepts ($C \sqcap D \sqsubseteq \bot$), by $\text{match}_{\equiv}(C, D)$ the exact match between the concepts ($C \equiv D$), by $\text{match}_{\sqsubseteq}(C, D)$ the plugin match between the concepts ($C \sqsubseteq D$), and by $\text{match}_{\sqsupseteq}(C, D)$ the subsume match between the concepts ($C \sqsupseteq D$). When we write $\text{match}_{\sqsubseteq\!\!\!\sqsupseteq}(C, D)$ we mean any one of the exact, plugin, or subsume match. The symbol $\sqsubseteq\!\!\!\sqsupseteq$ is a placeholder for ($\equiv$, $\sqsubseteq$, or $\sqsupseteq$).

4. Problem Formulation

There is a clear distinction between the types of semantic matches and results they provide, as well as the way the results present themselves w.r.t. some background knowledge. Intersection matches provide only information whether the concepts in the semantic match are compatible or not. For example, if two concepts are not compatible, the (negative) result of the intersection match between them presents as an entailment and to explain it we can invoke a justification, which will provide the reasons why the concepts are not compatible. However, if two concepts are, in fact, compatible, it presents as a non-entailment. To explain this (positive) result of their intersection match, we refer to exact, plugin, or subsume matches. To put it simply, the information that the positive intersection match gave can be further used to identify the degree to which the matching concepts *could be* semantically similar. If we explain the extent of their similarity, we also inherently explain why they are compatible.

Let $\mathcal{T}$ be an $\mathcal{EL}_\bot$ TBox and match$_{\underline{\sqsubset}}(C,D)$ a semantic match of higher degree (exact, plugin, or subsume), such that the outcome of the semantic match is negative, $\mathcal{T} \not\models \text{match}_{\underline{\sqsubset}}(C,D)$. A classical approach to explain this non-entailment is *abduction*, i.e. finding "missing" knowledge such that when added to $\mathcal{T}$ the result of the semantic match becomes positive. If the matching concepts should, in fact, be in a positive exact, plugin, or subsume semantic relation, i.e. if match$_{\underline{\sqsubset}}(C,D)$ should logically follow from $\mathcal{T}$, then abduction allows us to find reasons why it was not entailed and fix the non-entailment. The abduction problem we are interested in for semantic matching is defined as follows:

Definition 7. *Let $\mathcal{T}$ be an $\mathcal{EL}_\bot$ TBox and match$_{\underline{\sqsubset}}(C,D)$ an exact ($C \equiv D$), plugin ($C \sqsubseteq D$), or subsume ($C \sqsupseteq D$) semantic match between concepts C and D. An abduction problem is a tuple $\langle \mathcal{T}, \text{match}_{\underline{\sqsubset}}(C,D)\rangle$, where $\mathcal{T}$ is the background knowledge, $\mathcal{T} \not\models C \sqcap D \equiv \bot$ and $\mathcal{T} \not\models \text{match}_{\underline{\sqsubset}}(C,D)$, i.e. the result of the semantic match is negative (non-entailment). A solution to the abduction problem is a hypothesis $\mathcal{H}$ of the form:*

$$\mathcal{H} = \{\alpha \mid \mathcal{T} \not\models \alpha\},$$

such that $\mathcal{T} \cup \mathcal{H} \not\models C \sqcap D \equiv \bot$ and $\mathcal{T} \cup \mathcal{H} \models \text{match}_{\underline{\sqsubset}}(C,D)$.

5. Explaining Non-entailments of Semantic Matching

To explain non-entailments of semantic matching we opt to search for "missing" connections between concept descriptions introduced in a negative match. The case of intersection matching is used purely to identify whether the background knowledge can, in fact, provide a consistent model of a semantic relation between matching concepts. For the negative outcomes of higher degree matches, we need to construct hypotheses that will contain the axioms such that when added to the background knowledge the outcome of the match becomes positive. To do this, we need to take into account the following:

Point 1. Hypotheses should contain axioms that preserve the structure of concept descriptions (definitions) introduced in matches,
Point 2. Hypotheses should take into account the structure of concept descriptions that cannot be preserved,
Point 3. Hypotheses should not unnecessarily omit parts of concept descriptions that preserve the structure.

The idea of preserving the structure in **Point 1** refers to avoiding abduction of unrelated concepts. We present the notion of an $\mathcal{EL}$ *description tree*, as originally defined in [40] and revisited in [15], which is a graphical representation of $\mathcal{EL}$ concepts.

Definition 8. *Let $\mathcal{T}$ be an $\mathcal{EL}_\bot$ TBox with signature $\Sigma = \langle N_C, N_R\rangle$. An $\mathcal{EL}$ description tree is a node-labeled and edge-labeled directed rooted tree, $T = \langle \mathcal{V}, \mathcal{E}, v_0, \lambda_\mathcal{V}, \lambda_\mathcal{E}\rangle$, where:*

- *$\mathcal{V}$ is a finite set of nodes,*
- *$\mathcal{E}$ is a finite set of edges,*
- *v_0 is the root node,*

- $\lambda_{\mathcal{V}} : \mathcal{V} \mapsto L_{\mathcal{V}}$, *such that for any* $v \in \mathcal{V}$ *and* $l_{\mathcal{V}} \in L_{\mathcal{V}}$, $\lambda_{\mathcal{V}}(v) = l_{\mathcal{V}}$ *and* $l_{\mathcal{V}} \subseteq \mathcal{N}_{\mathcal{C}}$, *and*
- $\lambda_{\mathcal{E}} : \mathcal{E} \mapsto L_{\mathcal{E}}$, *such that for any* $e \in \mathcal{E}$ *and* $l_{\mathcal{E}} \in L_{\mathcal{E}}$, $\lambda_{\mathcal{E}}(e) = l_{\mathcal{E}}$ *and* $l_{\mathcal{E}} \in \mathcal{N}_{\mathcal{R}}$.

The empty label represents the top concept ($\top$). The bottom concept ($\bot$) is not allowed in description trees.

We can recursively define a concept C which is represented by a description tree. Let $T_C = \langle \mathcal{V}_C, \mathcal{E}_C, v_0, \lambda_{\mathcal{V}_C}, \lambda_{\mathcal{E}_C} \rangle$ be the description tree for the concept definition C. If $v_1, v_2, \ldots, v_n$ are the children of v_0 and we denote by $T_C(v_1), T_C(v_2), \ldots, T_C(v_n)$ the pairwise disjoint subtrees of T_C rooted in $v_1, v_2, \ldots, v_n$, then we can define the concept C as:

$$C = C_{T_C(v_0)}, \tag{9}$$

$$C_{T_C(v_i)} = \prod_{v_i \in \mathcal{V}} \lambda_{\mathcal{V}_C}(v_i) \sqcap \prod_{\langle v_i, v_j \rangle \in \mathcal{E}_C} \exists \lambda_{\mathcal{E}_C}(\langle v_i, v_j \rangle).C_{T_C(v_j)},$$

where $C_{T(v_0)}$ is the concept represented by the tree rooted in v_0, and $C_{T(v_i)}$ are the concepts represented by the trees rooted in v_i.

Likewise, if we have a concept definition $C \equiv \prod_{i=0}^{n} A_i \sqcap \exists r_1.C_1 \sqcap \cdots \sqcap \exists r_n.C_n$, we can define the description tree $T_C = \langle \mathcal{V}_C, \mathcal{E}_C, v_0, \lambda_{\mathcal{V}_C}, \lambda_{\mathcal{E}_C} \rangle$ of C inductively, based on the pairwise disjoint description trees $T_{C_1}, T_{C_2}, \ldots, T_{C_n}$ for concepts $C_1, C_2, \ldots, C_n$ in the definition of C. If we denote by $T_i = \langle \mathcal{V}_i, \mathcal{E}_i, v_i, \lambda_{\mathcal{V}_i}, \lambda_{\mathcal{E}_i} \rangle$ the description trees for concepts C_i, then:

$$\mathcal{V}_C = \{v_0\} \bigcup_{i=1}^{n} \mathcal{V}_i,\ \mathcal{E}_C = \{\langle v_0, v_i \rangle \mid 1 \leqslant i \leqslant n\} \bigcup_{i=1}^{n} \mathcal{E}_i,\ \lambda_{\mathcal{V}_C}(v_0) = \{A_0, A_1, \ldots, A_n\}, \tag{10}$$

$$\lambda_{\mathcal{V}_C}(v_i) = \lambda_{\mathcal{V}_i}(v_i), \text{for any } v_i \in \mathcal{V}_i,$$

$$\lambda_{\mathcal{E}_C}(\langle v_i, v_j \rangle) = \lambda_{\mathcal{E}_i}(\langle v_i, v_j \rangle), \text{for any } \langle v_i, v_j \rangle \in \mathcal{E}_i.$$

Representing concept definitions as description trees gives the potential to relate concepts w.r.t. a background knowledge. For example, homomorphisms between description trees capture subsumption between $\mathcal{EL}$ concepts [15, 40]. This is achieved by generating a mapping between the vertex sets of description trees which preserves the structure between edges and labels. Consequently, this structure preserving map translates in the form of a concept inclusion, w.r.t. a background knowledge, between the concepts represented by those description trees. We focus on description tree isomorphisms, as a means of capturing subsumption and equivalence relations between concept definitions in semantic matching.

Definition 9. *Let* $T_C = \langle \mathcal{V}_C, \mathcal{E}_C, v_0, \lambda_{\mathcal{V}_1}, \lambda_{\mathcal{E}_C} \rangle$ *and* $T_D = \langle \mathcal{V}_D, \mathcal{E}_D, w_0, \lambda_{\mathcal{V}_D}, \lambda_{\mathcal{E}_2} \rangle$ *be two description trees. A weak isomorphism from* T_C *to* T_D *is a bijective mapping* $\phi : T_C \mapsto T_D$, *such that:*

1. $\phi(v_0) = w_0$,
2. $\langle v_i, v_j \rangle \in \mathcal{E}_C \Leftrightarrow \langle \phi(v_i), \phi(v_j) \rangle \in \mathcal{E}_D$ *and* $\lambda_{\mathcal{E}_C}(\langle v_i, v_j \rangle) = \lambda_{\mathcal{E}_D}(\langle \phi(v_i), \phi(v_j) \rangle)$.

We write $T_C \cong T_D$ *if two trees are weakly isomorphic.*

We propagate the notation from [15], and distinguish description tree isomorphisms in two ways: weak isomorphisms, i.e. isomorphisms that satisfy the conditions in Definition 9, and $\mathcal{T}$-isomorphisms, which are weak isomorphisms that contain a semantic layer.

Definition 10. *Let* $\mathcal{T}$ *be an* $\mathcal{EL}_\bot$ *TBox and* $T_C = \langle \mathcal{V}_C, \mathcal{E}_C, v_0, \lambda_{\mathcal{V}_C}, \lambda_{\mathcal{E}_C} \rangle$ *and* $T_D = \langle \mathcal{V}_D, \mathcal{E}_D, w_0, \lambda_{\mathcal{V}_D}, \lambda_{\mathcal{E}_D} \rangle$ *be two description trees. A weak isomorphism* $\phi : T_C \mapsto T_D$ *becomes a* $\mathcal{T}$*-isomorphism, when* $\forall v \in \mathcal{V}_C$, *and* $w \in \mathcal{V}_D$ *s.t.* $\phi(v) = w$, *one of the following holds:*

1. $\mathcal{T} \models \prod_{v \in \mathcal{V}_C} \lambda_{\mathcal{V}_C}(v) \equiv \prod_{w \in \mathcal{V}_D} \lambda_{\mathcal{V}_C}(w)$, *or*

2. $\mathcal{T} \models \bigsqcap_{v \in \mathcal{V}_C} \lambda_{\mathcal{V}_C}(v) \sqsubseteq \bigsqcap_{w \in \mathcal{V}_D} \lambda_{\mathcal{V}_C}(w)$, *or*
3. $\mathcal{T} \models \bigsqcap_{v \in \mathcal{V}_C} \lambda_{\mathcal{V}_C}(v) \sqsupseteq \bigsqcap_{w \in \mathcal{V}_D} \lambda_{\mathcal{V}_C}(w)$.

The semantic layer in Definition 10 extends weak isomorphisms and allows for TBoxes to capture equivalence and subsumption, which is also convenient for capturing the relations in exact, plugin, and subsume matches.

Theorem 1. *Let $\mathcal{T}$ be an $\mathcal{EL}_\bot$ TBox and let C and D be two concepts defined w.r.t. $\mathcal{T}$, such that $T_C = \langle \mathcal{V}_C, \mathcal{E}_C, v_0, \lambda_{\mathcal{V}_C}, \lambda_{\mathcal{E}_C} \rangle$ and $T_D = \langle \mathcal{V}_D, \mathcal{E}_D, w_0, \lambda_{\mathcal{V}_D}, \lambda_{\mathcal{E}_D} \rangle$. If there exists a $\mathcal{T}$-isomorphism $\phi : T_C \mapsto T_D$, then either $\mathcal{T} \models C \equiv D$, $\mathcal{T} \models C \sqsubseteq D$, or $\mathcal{T} \models C \sqsupseteq D$ is true.*

Proof. The proof is done by structural induction. Assume $\phi : T_C \mapsto T_D$ to be a $\mathcal{T}$-isomorphism.

Base case: If $T_C = \langle \{v_0\}, \emptyset, v_0, \lambda_{\mathcal{V}_C}, \lambda_{\mathcal{E}_C} \rangle$ and $T_D = \langle \{w_0\}, \emptyset, w_0, \lambda_{\mathcal{V}_D}, \lambda_{\mathcal{E}_D} \rangle$, we have that $C \equiv \bigsqcap \lambda_{\mathcal{V}_C}(v_0)$ and $D \equiv \bigsqcap \lambda_{\mathcal{V}_D}(w_0)$. Since $\mathcal{T} \models \bigsqcap \lambda_{\mathcal{V}_C}(v_0) \equiv \bigsqcap \lambda_{\mathcal{V}_D}(w_0)$ and $\phi(v_0) = w_0$, it follows that $\mathcal{T} \models C \equiv D$.

Induction: We extend the trees $T_C = \langle \{v_0\} \cup \{v_i \mid 1 \leqslant i \leqslant n\}, \{\langle v_0, v_i \rangle \mid 1 \leqslant i \leqslant n\}, v_0, \lambda_{\mathcal{V}_C}, \lambda_{\mathcal{E}_C} \rangle$ and $T_D = \langle \{w_0\} \cup \{w_i \mid 1 \leqslant i \leqslant n\}, \{\langle w_0, w_i \rangle \mid 1 \leqslant i \leqslant n\}, w_0, \lambda_{\mathcal{V}_D}, \lambda_{\mathcal{E}_D} \rangle$, s.t. each v_i is the root of the subtree $T_C(v_i)$ in T_C and each w_i is the root of the subtree $T_D(w_i)$ in T_D. Since ϕ is a $\mathcal{T}$-isomorphism from T_C to T_D, for each child of v_0 in T_C there is a corresponding child of w_0 in T_D, s.t. $\phi(v_i) = w_i$.

Assuming that ϕ is also a $\mathcal{T}$-isomorphism from $T_C(v_i)$ to $T_D(w_i)$, by induction we have that $\mathcal{T} \models C_{T_C(v_i)} \equiv C_{T_D(w_i)}$ and subsequently $\mathcal{T} \models \exists \lambda_{\mathcal{E}_C}(\langle v_0, v_i \rangle).C_{T_C(v_i)} \equiv \exists \lambda_{\mathcal{E}_D}(\langle w_0, w_i \rangle).C_{T_D(w_i)}$ for all children of v_0 in T_C and w_0 in T_D. Because $\mathcal{T} \models \bigsqcap \lambda_{\mathcal{V}_C}(v_0) \equiv \bigsqcap \lambda_{\mathcal{V}_D}(w_0)$ we have that $\mathcal{T} \models C \equiv D$.

The proof for $\mathcal{T} \models C \sqsubseteq D$ and $\mathcal{T} \models C \sqsupseteq D$ is done similarly. □

Considering that $\mathcal{T}$-isomorphisms characterize concept equivalence and subsumption in $\mathcal{EL}_\bot$, they can be used to solve abduction problems as defined in Definition 7. Particularly, the concept inclusions from Definition 10 extend weak isomorphisms to $\mathcal{T}$-isomorphisms. Thus, we can search for weak isomorphisms between description trees, and, by adding a semantic layer in the form of a hypothesis we can extend them to, in fact, $\mathcal{T} \cup \mathcal{H}$-isomorphisms where needed. The hypothesis $\mathcal{H}$ will contain "missing" knowledge that explains the non-entailment. Dependent on the type of the semantic match we are interested in, we can perform abduction with either ($\equiv$) (to represent exact matches), ($\sqsubseteq$) (to represent a more specific match) or ($\sqsupseteq$) (to represent a less specific match). To illustrate how isomorphism capture concept relations consider the following example.

Example 1. *Let $\mathcal{T}$ be a TBox and let:*

$$C = A_0 \sqcap \exists r.A_1 \sqcap \exists s.A_2,$$
$$D = B_0 \sqcap \exists r.B_1 \sqcap \exists s.B_2$$

be two concept descriptions, such that $\mathcal{T} \not\models C \sqcap D \equiv \bot$ and $\mathcal{T} \not\models match_\sqsubseteq(C, D)$. The description trees of C and D are defined as follows:

$$\begin{aligned}
T_C = \langle \mathcal{V}_C &= \{v_0, v_1, v_2\}, \mathcal{E}_C = \{\langle v_0, v_1 \rangle, \langle v_0, v_2 \rangle\}, v_0, \\
\lambda_{\mathcal{V}_C} &= \{v_0 \mapsto \{A_0\}, v_1 \mapsto \{A_1\}, v_2 \mapsto \{A_2\}\}, \lambda_{\mathcal{E}_C} = \{\langle v_0, v_1 \rangle \mapsto r, \langle v_0, v_2 \rangle \mapsto s\}\rangle, \\
T_D = \langle \mathcal{V}_D &= \{w_0, w_1, w_2\}, \mathcal{E}_D = \{\langle w_0, w_1 \rangle, \langle w_0, w_2 \rangle\}, w_0, \\
\lambda_{\mathcal{V}_D} &= \{w_0 \mapsto \{B_0\}, w_1 \mapsto \{B_1\}, w_2 \mapsto \{B_2\}\}, \lambda_{\mathcal{E}_D} = \{\langle w_0, w_1 \rangle \mapsto r, \langle w_0, w_2 \rangle \mapsto s\}\rangle
\end{aligned}$$

and illustrated in Figure 1, along with a weak isomorphism $\phi = (v_0 w_0)(v_1 w_1)(v_2 w_2)$ between T_C and T_D shown in green double sided arrows. The following hypothesis:

$$\mathcal{H} = \{A_0 \sqsubseteq B_0, A_1 \sqsubseteq B_1, A_2 \sqsubseteq B_2\}$$

extends the weak isomorphism ϕ to a $\mathcal{T} \cup \mathcal{H}$-isomorphism, while preserving the structure between the description trees. It extends the weak isomorphism, and we have $\mathcal{T} \cup \mathcal{H} \not\models C \sqcap D \equiv \bot$ and $\mathcal{T} \cup \mathcal{H} \models match_{\sqsubseteq}(C, D)$.

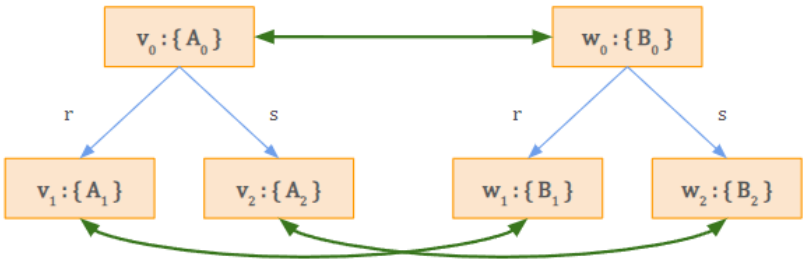

Fig. 1. Example description trees T_C (left) and T_D (right), for concepts C and D.

If relations between concepts are missing from some background knowledge $\mathcal{T}$, a weak isomorphism between the graphical representations of those concepts will point to which of those relations are missing. Thus, a semantic layer can be created on top using the weak isomorphism. This semantic layer is exactly the hypothesis that will explain the non-entailment. In other words the semantic layer will extend the weak isomorphism to a $\mathcal{T}$-isomorphism. In addition, weak isomorphisms and $\mathcal{T}$-isomorphisms preserve the structure between description trees of concepts and do not allow for unrelated concepts to be in hypotheses. With this **Point 1** is addressed. However, we need to extend the approach to be able to also abduct role restrictions.

Point 2 addresses the abduction of role restrictions. The key reason why a non-entailment exists may be a role restriction on a certain concept, that depicts a part of the structure of concept descriptions that cannot be preserved. Roles are presented as labels of edges in description trees. If the labels of edges between description trees are different, i.e. not preserved, then by Definition 9 those trees cannot be weakly isomorphic. This also means that the concepts in the descriptions cannot be related and a $\mathcal{T}$-isomorphism (or a $(\mathcal{T} \cup \mathcal{H})$-isomorphism) cannot be obtained. However, if we contract parts of descriptions trees, and, consequently of concept definitions, that are not preserved and introduce them in hypotheses, then a $(\mathcal{T} \cup \mathcal{H})$-isomorphism could still be reached. Since roles in concept definitions, which present as edge labels in descriptions trees, formulate the structure of the concepts, parts that cannot be preserved consist mainly of role restrictions. Thus, abduction of role restrictions is crucial when concepts' definitions cannot be preserved.

To allow for abduction of role restrictions we introduce subtree isomorphisms between $\rho_{\subseteq}$-subtrees.

Definition 11. *Let $T = \langle \mathcal{V}, \mathcal{E}, v_0, \lambda_{\mathcal{V}}, \lambda_{\mathcal{E}} \rangle$ be a description tree. Let $\mathcal{S} \subseteq \mathcal{V}$ be any subset of vertices of T. Then, the induced subtree $T[\mathcal{S}] = \langle \mathcal{S}, \mathcal{E}_{[\mathcal{S}]}, \rho_{\mathcal{S}}, \lambda_{[\mathcal{S}]}, \lambda_{\mathcal{E}_{[\mathcal{S}]}} \rangle$ is the subtree of T whose node set is $\mathcal{S}$, and:*

- *$T[\mathcal{S}]$ is rooted in $\rho_{\mathcal{S}}$,*
- *for any two nodes $v, u \in \mathcal{S}$, $\langle v, u \rangle \in \mathcal{E}_{[\mathcal{S}]}$ if $\langle v, u \rangle \in \mathcal{E}$,*
- *for all $v \in \mathcal{S}$, $\lambda_{[\mathcal{S}]}(v) = \lambda_{\mathcal{V}}(v)$,*
- *for all $e \in \mathcal{E}_{[\mathcal{S}]}$, $\lambda_{\mathcal{E}_{[\mathcal{S}]}}(e) = \lambda_{\mathcal{E}}(e)$.*

If $T[\mathcal{S}]$ is an induced subtree of T, we write $T[\mathcal{S}] \subseteq T$. $T[\mathcal{S}]$ is called a $\rho_{\subseteq}$-subtree if $T[\mathcal{S}]$ is an induced subtree of T and has the same root as T, $\rho_{\mathcal{S}} = v_0$, and denote it by $T[\mathcal{S}] \subseteq_{\rho} T$.

Observation 1. *A $\rho_{\subseteq}$-subtree is a description tree itself.*

Proof. This follows from the definition of $\rho_{\subseteq}$-subtrees. We directly prove this.

A $\rho_{\subseteq}$-subtree $T[\mathcal{S}] = \langle \mathcal{S}, \mathcal{E}_{[\mathcal{S}]}, v_0, \lambda_{[\mathcal{S}]}, \lambda_{\mathcal{E}_{[\mathcal{S}]}} \rangle$ of a description tree $T = \langle \mathcal{V}, \mathcal{E}, v_0, \lambda_{\mathcal{V}}, \lambda_{\mathcal{E}} \rangle$ has the same characteristics as a description tree.

- By definition $\mathcal{S} \subseteq \mathcal{V}$. Thus, $T[\mathcal{S}]$ has a finite set of vertices.
- If $\langle v, u\rangle \in \mathcal{E}$ for any two vertices $v, u \in \mathcal{S}$, then it is also true that $\langle v, u\rangle \in \mathcal{E}_{[\mathcal{S}]}$. Thus, $\mathcal{E}_{[\mathcal{S}]} \subseteq \mathcal{E}$ and $T[\mathcal{S}]$ has a finite set of edges.
- By definition of $\rho_{\subseteq}$-subtrees, $T[\mathcal{S}]$ has the same root as T.
- By definition, for all $v \in \mathcal{S}$ we have $\lambda_{[\mathcal{S}]}(v) = \lambda_{\mathcal{V}}(v)$. Since $\lambda_{\mathcal{V}}(v) \subseteq N_C$, we also have $\lambda_{[\mathcal{S}]}(v) \subseteq N_C$.
- By definition, for all $e \in \mathcal{E}_{[\mathcal{S}]}$ we have $\lambda_{\mathcal{E}_{[\mathcal{S}]}}(e) = \lambda_{\mathcal{E}}(e)$. Since $\lambda_{\mathcal{E}}(e) \in N_R$, we also have $\lambda_{\mathcal{E}_{[\mathcal{S}]}}(e) \in N_R$.

Thus, a $\rho_{\subseteq}$-subtree is a description tree itself. □

Observation 2. *Every description tree is a $\rho_{\subseteq}$-subtree of itself.*

Proof. The proof directly follows the definition of an induced subtree. We want to show for a description tree $T = \langle \mathcal{V}, \mathcal{E}, v_0, \lambda_{\mathcal{V}}, \lambda_{\mathcal{E}}\rangle$ that $T \subseteq_{\rho} T$. Then, T is the induced subtree of T, by taking all vertices of T as the inducing subset $\mathcal{S}$, $\mathcal{S} = \mathcal{V}$. Since, the root v_0 is the same, $T \subseteq_{\rho} T$. □

Because $\rho_{\subseteq}$-subtrees are description trees, definition 9 refers to weak isomorphisms between $\rho_{\subseteq}$-subtrees as well. In addition, we can extend weak isomorphisms to $\mathcal{T}$-isomorphisms for $\rho_{\subseteq}$-subtrees, thus providing a semantic layer. In order to do this, we need to take into account parts of descriptions trees that are not included in the weak isomorphisms.

Definition 12. *Let $\mathcal{T}$ be an $\mathcal{EL}_{\bot}$ TBox and $T_1 = \langle \mathcal{V}_1, \mathcal{E}_1, v_0, \lambda_{\mathcal{V}_1}, \lambda_{\mathcal{E}_1}\rangle$ and $T_2 = \langle \mathcal{V}_2, \mathcal{E}_2, w_0, \lambda_{\mathcal{V}_2}, \lambda_{\mathcal{E}_2}\rangle$ be two description trees. A weak isomorphism $\phi : T_1[\mathcal{S}_1] \mapsto T_2[\mathcal{S}_2]$ for $T_1[\mathcal{S}_1] \subseteq_{\rho} T_1$ and $T_2[\mathcal{S}_2] \subseteq_{\rho} T_2$ becomes a $\mathcal{T}$-isomorphism, when $\forall v \in \mathcal{S}_1$, and $w \in \mathcal{S}_2$ s.t. $\phi(v) = w$, one of the following holds:*

1. $\mathcal{T} \models \lambda^*_{[\mathcal{S}_1]}(v) \equiv \lambda^*_{[\mathcal{S}_2]}(w)$, *or*
2. $\mathcal{T} \models \lambda^*_{[\mathcal{S}_1]}(v) \sqsubseteq \lambda^*_{[\mathcal{S}_2]}(w)$, *or*
3. $\mathcal{T} \models \lambda^*_{[\mathcal{S}_1]}(v) \sqsupseteq \lambda^*_{[\mathcal{S}_2]}(w)$,

where:

$$\lambda^*_{[\mathcal{S}_1]}(v) = \bigsqcap_{v \in \mathcal{S}_1} \lambda_{[\mathcal{S}_1]}(v) \sqcap \bigsqcap_{\langle v,x\rangle \in \mathcal{E}_1, x \notin \mathcal{S}_1} \exists \lambda_{\mathcal{E}_1}(\langle v, x\rangle).C_{T_1(x)}, \tag{11}$$

and

$$\lambda^*_{[\mathcal{S}_2]}(w) = \bigsqcap_{w \in \mathcal{S}_2} \lambda_{[\mathcal{S}_2]}(w) \sqcap \bigsqcap_{\langle w,y\rangle \in \mathcal{E}_2, y \notin \mathcal{S}_2} \exists \lambda_{\mathcal{E}_2}(\langle w, y\rangle).C_{T_2(y)}, \tag{12}$$

Definition 12 states that for a given isomorphism between two $\rho_{\subseteq}$-subtrees, the vertices and edges that have induced a subtree will adjust the labels that provide the semantic layer. This adjustment allows for abduction of complex concept expressions which include concepts, role restrictions, or a combination of both.

Example 2. *To exemplify a subtree isomorphism between description trees and abduction of concepts and roles, let $\mathcal{T}$ be a TBox and $C = A_0 \sqcap \exists r.A_1 \exists s.A_2 \sqcap \exists p.A_3$ and $D = B_0 \sqcap \exists r.B_1 \sqcap \exists s.B_2$ concepts, such that $\mathcal{T} \not\models C \sqsubseteq D$. The description trees of C and D are shown in Figure 2. It is apparent that the description trees T_C and T_D are not isomorphic. However, we can find subparts of T_C that are isomorphic to T_D. If we consider the following $\rho_{\subseteq}$-subtree of T_C:*

$$\begin{aligned} T_C[\mathcal{S}_C] =& \langle \mathcal{S}_C, \mathcal{E}_{[\mathcal{S}_C]}, v_0, \lambda_{[\mathcal{S}_C]}, \lambda_{\mathcal{E}_{[\mathcal{S}_C]}}\rangle = \\ & \langle \mathcal{S}_C = \{v_0, v_1, v_2\}, \mathcal{E}_{[\mathcal{S}_C]} = \{\langle v_0, v_1\rangle, \langle v_0, v_2\rangle\}, v_0, \\ & \lambda_{[\mathcal{S}_C]} = \{v_0 \mapsto \{A_0\}, v_1 \mapsto \{A_1\}, v_2 \mapsto \{A_2\}, \\ & \lambda_{\mathcal{E}_{[\mathcal{S}_C]}} = \{\langle v_0, v_1\rangle \mapsto r, \langle v_0, v_2\rangle \mapsto s\}\rangle \end{aligned} \tag{13}$$

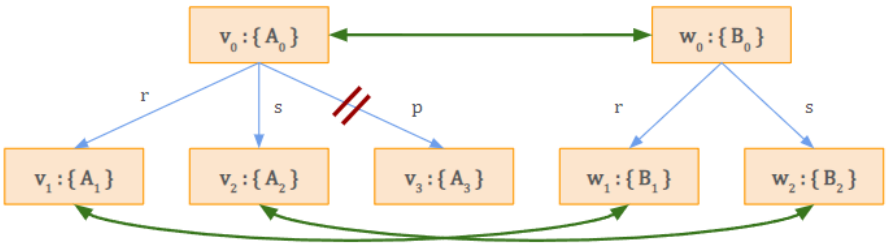

Fig. 2. Description trees T_C (left) and T_D (right) and a subtree isomorphism.

of T_C, we can find a weak isomorphism $\phi : T_C[\mathcal{S}_C] \mapsto T_D$, such that $\phi = (v_0w_0)(v_1w_1)(v_2w_2)$. Using this weak isomorphism we can construct the hypothesis $\mathcal{H} = \{A_0 \sqcap \exists p.A_3 \sqsubseteq B_0, A_1 \sqsubseteq B_1, A_2 \sqsubseteq B_2\}$ which extends it to a $\mathcal{T} \cup \mathcal{H}$-isomorphism. We now have $\mathcal{T} \cup \mathcal{H} \models C \sqsubseteq D$ and an explanation for the non-entailment.

The final **Point 3** addresses the parsimony of the abduction process. Naturally, hypotheses that contain more complex expressions are harder to interpret in comparison to those containing simpler abductions. Moreover, for an abduction problem defined as in Definition 7, any arbitrary abduction up to the entire descriptions of the matching concepts can be used to explain the non-entailment. To this end, we introduce a minimality criteria which filters out hypotheses that contain unnecessary abductions of concepts and/or role restrictions.

To define minimal abductive solutions, we first impose a partial order on the isomorphisms.

Definition 13. *Let $T_1 = \langle \mathcal{V}_1, \mathcal{E}_1, v_0, \lambda_{\mathcal{V}_1}, \lambda_{\mathcal{E}_1} \rangle$ and $T_2 = \langle \mathcal{V}_2, \mathcal{E}_2, w_0, \lambda_{\mathcal{V}_2}, \lambda_{\mathcal{E}_2} \rangle$ be two description trees with $T_1[\mathcal{S}_1] \subseteq_\rho T_1$, $T_1[\mathcal{S}'_1] \subseteq_\rho T_1$, $T_2[\mathcal{S}_2] \subseteq_\rho T_2$, and $T_2[\mathcal{S}'_2] \subseteq_\rho T_2$. If $\phi : T_1[S_1] \mapsto T_1[S_2]$ and $\phi' : T_1[S'_1] \mapsto T_2[S'_2]$, then $\phi \preceq \phi'$ if:*

$$|\mathcal{S}_1| + |\mathcal{S}_2| \geqslant |\mathcal{S}'_1| + |\mathcal{S}'_2| \tag{14}$$

Definition 13 partially orders weak isomorphisms between $\rho_{\subseteq}$-subtrees, such that, the minimal element of the partially ordered set is the weak isomorphism obtained between largest $\rho^{\subseteq}$-subtrees. From this, we gain information on how much the structure has changed in order to find isomorphic $\rho^{\subseteq}$-subtrees of description trees. The more changes to the structure, the more complex restrictions are included in the hypotheses, making them more difficult to interpret. In cases where there are more than one isomorphic mappings, the minimal one is preferred. As a result of the hypotheses being constructed from weak isomorphisms, the partial order directly appoints minimal abductive solutions, i.e. hypotheses.

Definition 14. *Given an abduction problem $(\mathcal{T}, match_{\underline{\sqsubseteq}}(C,D))$, with $T_C = \langle \mathcal{V}_C, \mathcal{E}_C, v_0, \lambda_{\mathcal{V}_C}, \lambda_{\mathcal{E}_C} \rangle$ and $T_D = \langle \mathcal{V}_D, \mathcal{E}_D, w_0, \lambda_{\mathcal{V}_D}, \lambda_{\mathcal{E}_D} \rangle$ which represent the description trees of C and D, respectively, a solution $\mathcal{H}$ obtained from a weak isomorphism ϕ is considered minimal if there does not exist any $\mathcal{H}'$, obtained from ϕ', such that $\phi' \prec \phi$. We say that $\mathcal{H} \prec \mathcal{H}'$ if $\phi \prec \phi'$.*

We prioritize preserving the concept definitions, and, where necessary, find solutions obtained from minimal number of contractions to induce subtrees. To illustrate minimal solutions consider the following example.

Example 3. *Let $\mathcal{T}$ be a TBox and let C and D be two concept descriptions defined w.r.t. the signature of $\mathcal{T}$, such that $\mathcal{T} \not\models C \sqcap D \equiv \bot$ and $\mathcal{T} \not\models match_{\sqsubseteq}(C,D)$, where:*

$$C = A_0 \sqcap \exists r.A_1 \sqcap \exists s.A_2,$$

$$D = B_0 \sqcap \exists r.B_1 \sqcap \exists s.B_2.$$

The descriptions trees of the concepts C and D are shown in Figure 1. We can construct more than one hypotheses to explain this non-entailment. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two hypotheses, such that:

$$\mathcal{H}_1 = \{A_0 \sqsubseteq B_0, A_1 \sqsubseteq B_1, A_2 \sqsubseteq B_2\},$$
$$\mathcal{H}_2 = \{A_0 \sqcap \exists r.A_1 \sqsubseteq B_0 \sqcap \exists r.B_1, A_2 \sqsubseteq B_2\}$$

Both $\mathcal{H}_1$ and $\mathcal{H}_2$ explain the non-entailment and we have $\mathcal{T} \cup \mathcal{H}_1 \models C \sqsubseteq D$ and $\mathcal{T} \cup \mathcal{H}_2 \models C \sqsubseteq D$. However, $\mathcal{H}_2$ is constructed by performing unnecessary contractions in the description trees, and, consequently the concept definitions, that only introduces more complex expressions to interpret in the hypothesis. In fact, we can go up to any arbitrary number of contractions that induce subtrees and even construct a hypothesis $\mathcal{H}_n = \{A_0 \sqcap \exists r.A_1 \sqcap \exists s.A_2 \sqsubseteq B_0 \sqcap \exists r.B_1 \sqcap \exists s.B_2\}$, which includes the complete definitions of the concepts C and D. However, this explanation does not provide any meaningful relations that could be used to explain the non-entailment and update the terminological knowledge. To this end, we filter the hypotheses w.r.t. the minimality criteria in Definition 14.

Isomorphisms between $\rho_{\subseteq}$-subtrees allow for relating relevant concepts and role restrictions whilst preserving the structure of concept definitions. Furthermore, they do not restrict the process of abduction to a predetermined set of abducibles, but they allow for any form of a complex expression to be abducted. The only limitation is w.r.t. the signature of a background knowledge for which we want to explain a certain non-entailment. Moreover, the introduced minimality criteria filters hypotheses with arbitrary abductions and produces meaningful explanations.

5.1. Computing Subtree Isomorphisms

One of the most fundamental computational problems on trees is the subtree isomorphism problem, which asks whether a given tree is contained in another given tree. The subtree isomorphism problem has a few variants, which are mainly dependent on whether the trees are rooted or unrooted, whether the degrees of nodes are bounded, and whether an order on their nodes must be preserved. We accustom the subtree isomorphism problem for the special case of description trees to solve abduction problems in $\mathcal{EL}_\bot$.

Definition 15. *Let $T_1 = \langle \mathcal{V}_1, \mathcal{E}_1, v_0, \lambda_{\mathcal{V}_1}, \lambda_{\mathcal{E}_1} \rangle$ and $T_2 = \langle \mathcal{V}_2, \mathcal{E}_2, w_0, \lambda_{\mathcal{V}_2}, \lambda_{\mathcal{E}_2} \rangle$ be two description trees. A $\rho_{\subseteq}$-subtree isomorphism is an isomorphism between a $\rho_{\subseteq}$-subtree $T_1[\mathcal{S}_1] = \langle \mathcal{S}_1, \mathcal{E}_{\mathcal{S}_1}, v_0, \lambda_{[\mathcal{S}_1]}, \lambda_{\mathcal{E}_{\mathcal{S}_1}} \rangle$ of T_1 and a $\rho_{\subseteq}$-subtree $T_2[\mathcal{S}_2] = \langle \mathcal{S}_2, \mathcal{E}_{\mathcal{S}_2}, w_0, \lambda_{\mathcal{S}_2}, \lambda_{\mathcal{E}_{\mathcal{S}_2}} \rangle$ of T_2, such that $T_1[\mathcal{S}_1] \cong T_2[\mathcal{S}_2]$.*

To compute subtree isomorphisms between description trees and derive hypotheses to explain non-entailments of semantic matching we partition the children of vertices in their respective trees w.r.t. the labels of edges connecting parents to their children. We define an *equivalence relation* on the set of children for a vertex v in a description tree.

Theorem 2. *Let $\sim$ be a relation on a non-empty set of children, $N(v)$, for a node $v \in \mathcal{V}$ of a tree $T = \langle \mathcal{V}, \mathcal{E}, v_0, \lambda_{\mathcal{V}}, \lambda_{\mathcal{E}} \rangle$, such that for $x, y \in N(v)$, $x \sim y$ iff $\langle v, x \rangle \in \mathcal{E}$, $\langle v, y \rangle \in \mathcal{E}$ and $\lambda_{\mathcal{E}}(\langle v, x \rangle) = \lambda_{\mathcal{E}}(\langle v, y \rangle)$. Then, $\sim$ is an equivalence relation.*

Proof. For $\sim$ to be an equivalence relation we need to show that it is reflexive, symmetric, and transitive. Let $T = \langle \mathcal{V}, \mathcal{E}, v_0, \lambda_{\mathcal{V}}, \lambda_{\mathcal{E}} \rangle$ and $v \in \mathcal{V}$ an arbitrary node.

Reflexive property: For $x \in N(v)$ and $\langle v, x \rangle \in \mathcal{E}$, we have that $\lambda_{\mathcal{E}}(\langle v, x \rangle) = \lambda_{\mathcal{E}}(\langle v, x \rangle)$.

Symmetric property: For $x, y \in N(v)$, $\langle v, x \rangle \in \mathcal{E}$, and $\langle v, y \rangle \in \mathcal{E}$, if $\lambda_{\mathcal{E}}(\langle v, x \rangle) = \lambda_{\mathcal{E}}(\langle v, y \rangle)$, then $\lambda_{\mathcal{E}}(\langle v, y \rangle) = \lambda_{\mathcal{E}}(\langle v, x \rangle)$.

Transitive property: For $x, y, z \in N(v)$, $\langle v, x \rangle \in \mathcal{E}$, $\langle v, y \rangle \in \mathcal{E}$, and $\langle v, z \rangle \in \mathcal{E}$, if $\lambda_{\mathcal{E}}(\langle v, x \rangle) = \lambda_{\mathcal{E}}(\langle v, y \rangle)$ and $\lambda_{\mathcal{E}}(\langle v, y \rangle) = \lambda_{\mathcal{E}}(\langle v, z \rangle)$, then $\lambda_{\mathcal{E}}(\langle v, x \rangle) = \lambda_{\mathcal{E}}(\langle v, z \rangle)$.

Thus, $\sim$ is an equivalence relation. □

An equivalence relation defines exclusive classes whose members bear the relation to each other and not to those in other classes. The set of all equvalence classes is defined as:

$$N(v)_{/\sim} = \{[x] \mid x \in N(v)\}, \text{ where } [x] := \{y \in N(v) \mid x \sim y\}, \tag{15}$$

is the equivalence class of x.

Theorem 3. *Let $\sim$ be an equivalence relation on a non-empty set of neighbors $N(v)$, for a node $v \in \mathcal{V}$ of a tree $T = \langle \mathcal{V}, \mathcal{E}, v_0, \lambda_\mathcal{V}, \lambda_\mathcal{E} \rangle$. The collection of equivalence classes under $\sim$ forms a partition of $N(v)$.*

Proof. Let $\sim$ be an equivalence relation on $N(v)$ and let $x \in N(v)$.

1) We need to show that $N(v) = \bigcup_x [x]$, where $[x]$ is termed a *cell* (a subset of $N(v)$). If $x \in N(v)$, then x belongs to $[x]$ (by the reflexive property we have that $x \sim x$, therefore $x \in [x]$). Since this is true for each element of $N(v)$, we have that $N(v) = \bigcup_x [x]$.

2) We need to show that if $[x]$ and $[y]$ are any two cells, then either $[x] = [y]$ or $[x] \cap [y] = \emptyset$ holds.

2.1) Let $\sim$ be an equivalence relation defined on $N(v)$ with $x, y \in N(v)$ and let $x \in [y]$. Then $[x] = [y]$. If $x \in [y]$ then by definition of equivalence classes we have $x \sim y$. First we show that $[x] \subseteq [y]$. Let $a \in [x]$. By definition of equivalence classes, $a \sim x$, and by the transitive property we have that $a \sim y$. Therefore, $a \in [y]$ and $[x] \subseteq [y]$. Next, we show that $[y] \subseteq [x]$. If $b \in [y]$, then $b \sim y$. By symmetry we have that $y \sim x$ and by transitivity that $b \sim x$. Thus, $b \in [x]$ and $[y] \subseteq [x]$. Thus we have that $[x] = [y]$ if $x \in [y]$.

2.2) Suppose $[x] \cap [y] \neq \emptyset$. Then, there must be some element a, such that $a \in [x]$ and $a \in [y]$. Then, $[a] = [x]$ and $[a] = [y]$. Thus, $[x] = [y]$. □

Once the children are partitioned, vertices that belong in an equivalence class in one tree can be mapped to vertices that belong in an equivalence class in the other tree, such that the equivalence classes in the respective trees are obtained from same edge labels. Let $T_1 = \langle \mathcal{V}_1, \mathcal{E}_1, v_0, \lambda_{\mathcal{V}_1}, \lambda_{\mathcal{E}_1} \rangle$ and $T_2 = \langle \mathcal{V}_2, \mathcal{E}_2, w_0, \lambda_{\mathcal{V}_2}, \lambda_{\mathcal{E}_2} \rangle$ be two description trees. We define the relation:

$$\mu(v, w) \subseteq N(v)_{/\sim} \times N(w)_{/\sim}, \tag{16}$$

for nodes $v \in \mathcal{V}_1$ and $w \in \mathcal{V}_2$, such that:

$$\mu(v, w) := \{\langle [x], [y] \rangle \mid x \in N(v), y \in N(w), \langle v, x \rangle \in \mathcal{E}_1, \langle w, y \rangle \in \mathcal{E}_2 \text{ and } \xi_1(\langle v, x \rangle) = \xi_2(\langle w, y \rangle)\} \tag{17}$$

Finally, we map the partitioned nodes from the respective trees using the relation $\mu(v, w)$. This is done for all vertices that preserve the structure between the trees, whilst the vertices from partitions that cannot be mapped are contracted, thus inducing a subtree.

Let A and B be arbitrary sets, such that $|A| \leqslant |B|$. We define the injective function between A and B as $i : A \overset{1-1}{\mapsto} B$, such that for $a, b \in A$, $i(a) \neq i(b) \implies a \neq b$. We use the notation $A^{\underline{B}}$ for the set of all injections from A to B. Thus, we denote the set of all injective mappings between equivalence classes by $[x]^{\underline{[y]}}$ for $|[x]| \leqslant |[y]|$. For each pair of equivalence classes obtained from the equivalence relation, we find the set of all injective maps, such that:

$$I(v, w) = \{i \mid i = [x]^{\underline{[y]}}, \text{if } |[x]| \leqslant |[y]|, \text{ otherwise } i = [y]^{\underline{[x]}}, \text{ for } \langle [x], [y] \rangle \in \mu(v, w)\} \tag{18}$$

Each set of injections is unique to the respective pair of equivalence classes. To obtain all combinations of mapped vertices that produce isomorphisms, we search for the Cartesian product between the distinct sets of injections:

$$\mathcal{M}(v, w) = \prod_{i \in I(v,w)} i \tag{19}$$

To obtain isomorphic mappings, we construct a *Solutions Tree*.

Definition 16. *A Solutions Tree is a node-labeled rooted tree $T_\Phi = \langle \mathcal{V}_\Phi, \mathcal{E}_\Phi, \phi_0, \lambda_\Phi \rangle$ that contains vertex mappings between two description trees $T_1 = \langle \mathcal{V}_1, \mathcal{E}_1, v_0, \lambda_{\mathcal{V}_1}, \lambda_{\mathcal{E}_1} \rangle$ and $T_2 = \langle \mathcal{V}_2, \mathcal{E}_2, w_0, \lambda_{\mathcal{V}_2}, \lambda_{\mathcal{E}_2} \rangle$. The Solutions Tree contains nodes $\phi \in \mathcal{V}_\Phi$, such that:*

$$\lambda_\Phi(\phi) = \{\langle v, w \rangle \mid v \in \mathcal{V}_1, w \in \mathcal{V}_2\} \tag{20}$$

Each unique path from the root to a leaf node in the solutions tree contains a unique subtree isomorphism between T_1 and T_2. The isomorphisms are returned from the set of complete paths in the solutions tree.

The solutions tree is a rooted tree, which contains nodes that carry mapped vertices between two $\rho_{\subseteq}$-subtrees. Each unique path in the solutions tree provides a unique combination of mapped vertices between isomorphic $\rho_{\subseteq}$-subtrees of some description trees. We inductively define the sets $\mathcal{V}_\Phi$ and $\mathcal{E}_\Phi$ and the node labels of the solutions tree.

$$\begin{aligned}
&\textbf{B1}\ T_\Phi = \langle \mathcal{V}_\Phi, \mathcal{E}_\Phi, \phi_0, \lambda_\Phi \rangle, \text{ where } \mathcal{V}_\Phi = \{\phi_0\}, \mathcal{E}_\Phi = \emptyset, \lambda_\Phi = \{\phi_0 \mapsto \{\langle v_0, w_0 \rangle\}\} \\
&\textbf{R1}\ \text{if } \phi \in \mathcal{V}_\Phi, \text{ then for all } m \in \prod_{\langle v,w\rangle \in \lambda_\Phi(\phi)} \mathcal{M}(v,w), \phi^* \in \mathcal{V}_\Phi \text{ and } \langle \phi, \phi^* \rangle \in \mathcal{E}_\Phi \text{ where } \lambda_\Phi(\phi^*) = m. \\
&\textbf{R2}\ \text{Nothing else is in } \mathcal{V}_\Phi \text{ and } \mathcal{E}_\Phi.
\end{aligned} \tag{21}$$

The set of all isomorphic mappings Φ is then obtained from the set of all complete paths Π^0 in the solutions tree T_Φ as follows:

$$\Phi = \{\bigcup_{\phi \in \pi} \lambda_\Phi(\phi) \mid \pi \in \Pi^0\} \tag{22}$$

Claim 1. *If $\phi \in \Phi$, then the following is true: $P(\phi) := \phi$ is a weak isomorphism, $\phi : T_1[\mathcal{S}_1] \mapsto T_2[\mathcal{S}_2]$, for $T_1[\mathcal{S}_1] \subseteq_\rho T_1$ and $T_2[\mathcal{S}_2] \subseteq_\rho T_2$.*

Proof. We prove this by structural induction.

Let $T_1 = \langle \mathcal{V}_1, \mathcal{E}_1, v_0, \lambda_{\mathcal{V}_1}, \lambda_{\mathcal{E}_1} \rangle$ and $T_2 = \langle \mathcal{V}_2, \mathcal{E}_2, w_0, \lambda_{\mathcal{V}_2}, \lambda_{\mathcal{E}_2} \rangle$ be two description trees.

Basis: $T_\Phi = \langle \mathcal{V}_\Phi, \mathcal{E}_\Phi, \phi_0, \lambda_\Phi \rangle$, where $\mathcal{V}_\Phi = \{\phi_0\}, \mathcal{E}_\Phi = \emptyset, \lambda_\Phi = \{\phi_0 \mapsto \{\langle v_0, w_0 \rangle\}\}$. The set of all complete paths in the solutions tree is $\Pi^0 = \{\langle \phi_0 \rangle\}$, from which we get a mapping $\phi = \{\langle v_0, w_0 \rangle\} \in \Phi$, such that $\phi(v_0) = w_0$. Thus, there exist

$$\begin{aligned}
&T_1[\mathcal{S}_1] = \langle \mathcal{S}_1 = \{v_0\}, \mathcal{E}_{[\mathcal{S}_1]} = \emptyset, v_0, \lambda_{[\mathcal{S}_1]}, \lambda_{\mathcal{E}_{[\mathcal{S}_1]}} \rangle, \text{ and} \\
&T_2[\mathcal{S}_2] = \langle \mathcal{S}_2 = \{w_0\}, \mathcal{E}_{[\mathcal{S}_2]} = \emptyset, w_0, \lambda_{[\mathcal{S}_2]}, \lambda_{\mathcal{E}_{[\mathcal{S}_2]}} \rangle
\end{aligned} \tag{23}$$

for which $P(\phi)$ is true.

Induction: Assume for some arbitrary $\phi \in \Phi$ that $P(\phi)$ is true for some $T_1[\mathcal{S}_1] = \langle \mathcal{S}_1, \mathcal{E}_{[\mathcal{S}_1]}, v_0, \lambda_{[\mathcal{S}_1]}, \lambda_{\mathcal{E}_{[\mathcal{S}_1]}} \rangle$ and $T_2[\mathcal{S}_2] = \langle \mathcal{S}_2, \mathcal{E}_{[\mathcal{S}_2]}, v_0, \lambda_{[\mathcal{S}_2]}, \lambda_{\mathcal{E}_{[\mathcal{S}_2]}} \rangle$, $T_1[\mathcal{S}_1] \subseteq_\rho T_1$ and $T_2[\mathcal{S}_2] \subseteq_\rho T_2$. We need to show that $P(\phi \cup m)$ is true, for $m \in \mathcal{M}(v,w)$. To do this, we need to follow the construction of m. We have two cases:

1. If $m = \emptyset$, then for $\langle v, w \rangle \in \phi$, $v \in \mathcal{V}_1$, $w \in \mathcal{V}_2$, $P(\phi \cup m) = P(\phi)$. By the inductive hypothesis, $P(\phi)$ is true.

2. If $m \neq \emptyset$, then for $\langle v, w \rangle \in \phi$ we get $\mathcal{M}(v,w)$ from $\mu(v,w)$ and $I(v,w)$, such that for each $m \in \mathcal{M}(v,w)$ and for each $\langle x, y \rangle \in m$, we have $\lambda_{\mathcal{E}_1}(\langle v, x \rangle) = \lambda_{\mathcal{E}_2}(\langle w, y \rangle)$. Hence, all vertices $u \in \mathcal{V}_1$ and $q \in \mathcal{V}_2$, for which $\lambda_{\mathcal{E}_1}(\langle v, u \rangle) \neq \lambda_{\mathcal{E}_2}(\langle w, q \rangle)$, are omitted from each $m \in \mathcal{M}(v,w)$ $\rho_{\subseteq}$-subtrees are induced from those vertices, $T_1[\mathcal{S}_1] \subseteq_\rho T_1$ and $T_2[\mathcal{S}_2] \subseteq_\rho T_2$.

Essentially, m is a mapping $m : \mathcal{S}_1' \mapsto \mathcal{S}_2'$, where for all $x \in \mathcal{S}_1'$ and $y \in \mathcal{S}_2'$ such that $m(x) = y$, we have $\lambda_{\mathcal{E}_1}(\langle v, x \rangle) = \lambda_{\mathcal{E}_2}(\langle w, y \rangle)$, where $v \in S_1$ and $w \in S_2$ (condition 2 of Definition 9 for weak isomorphisms is satisfied). Since $m \in \mathcal{M}(v,w)$ contains children of v and 2, m is an extension of ϕ, and we have $\phi \cup m : \mathcal{S}_1 \cup \mathcal{S}_1' \mapsto \mathcal{S}_2 \cup \mathcal{S}_2'$, and for all $(\phi \cup m)(v) = w$ and $(\phi \cup m)(v') = w'$, such that $\langle v, v' \rangle \in \mathcal{E}_1$ and $\langle w, w' \rangle \in \mathcal{E}_2$, $\lambda_{\mathcal{E}_1}(\langle v, v' \rangle) = \lambda_{\mathcal{E}_2}(\langle w, w' \rangle)$.

For $P(\phi \cup m)$ to be true it remains condition 1 of Definition 9 for weak isomorphisms to be satisfied. By the inductive hypothesis, $P(\phi)$ is true, thus it contains a mapping $\phi(v_0) = w_0$. Since m is an extension of ϕ, we have that $(\phi \cup m)(v_0) = w_0$.

Since both conditions for a weak isomorphism are satisfied, $P(\phi \cup m)$ is true.

□

For two given $\rho_{\subseteq}$-subtrees $T_1[\mathcal{S}_1] = \langle \mathcal{S}_1, \mathcal{E}_{[\mathcal{S}_1]}, v_0, \lambda_{\mathcal{S}_1}, \lambda_{\mathcal{E}_{[\mathcal{S}_1]}} \rangle$ and $T_2[\mathcal{S}_2] = \langle \mathcal{S}_2, \mathcal{E}_{[\mathcal{S}_2]}, w_0, \lambda_{\mathcal{S}_2}, \lambda_{\mathcal{E}_{[\mathcal{S}_2]}} \rangle$, the hypotheses are then formulated from the set of all isomorphic mappings Φ in the following way: for both trees find the vertices that have been contracted and update the vertex labels w.r.t. Definition 12. For example, for a given label of an arbitrary node x in a $\rho_{\subseteq}$-subtree $T[\mathcal{S}] = \langle \mathcal{S}, \mathcal{E}_{[\mathcal{S}]}, v_0, \lambda_{[\mathcal{S}]}, \lambda_{\mathcal{E}_{[\mathcal{S}]}} \rangle$, the new label of x is:

$$\lambda^*_{[\mathcal{S}]}(x) = \bigsqcap_{x \in \mathcal{S}} \lambda_{[\mathcal{S}]}(x) \sqcap \bigsqcap_{\langle x,y \rangle \in \mathcal{E}, y \notin \mathcal{S}} \exists \lambda_{\mathcal{E}}(\langle x, y \rangle).C_{T(y)}, \tag{24}$$

Finally, for some $\rho_{\subseteq}$-subtree isomorphisms in Φ, the hypothesis $\mathcal{H}$ is constructed as follows:

$$\mathcal{H} = \{\lambda^*_{[\mathcal{S}_1]}(v) \sqsubseteq \lambda^*_{[\mathcal{S}_2]}(w) \mid \langle v, w \rangle \in \phi, \phi \in \Phi\}, \tag{25}$$

where $v \in \mathcal{S}_1$ and $w \in \mathcal{S}_2$.

6. Implementation and Working Example

In this section we present the implementation of our method for explaining non-entailments of semantic matching in $\mathcal{EL}_\bot$, described in Section 5. We illustrate the implementation using a working example and present results from simulated scenarios.

6.1. Implementation

Currently, the algorithm is implemented in Java using the OWL API [55]. The classification and entailment checking is performed by the Pellet reasoner [56].

The input to the algorithm is a non-entailment of a certain semantic match. More specifically, the algorithm receives some background knowledge ($\mathcal{T}$), matching concept descriptions (C and D), and a type of a match (exact ($\equiv$), plugin ($\sqsubseteq$), or subsume $\sqsupseteq$), which is an abduction problem of the form defined in Definition 7. The proposed method is implemented in four main steps:

Step 1. Read input parameters for the abduction problem (background knowledge $\mathcal{T}$, concept descriptions C and D, and type of match $\sqsubseteq$).
Step 2. Get the description trees T_C and T_D for the inputted concept descriptions C and D w.r.t. Definition 8.
Step 3. Compute the set Φ of $\rho_{\subseteq}$-subtree isomorphisms between T_C and T_D obtained in **Step 2** w.r.t. proposed methodology in Section 5.
Step 4. Construct the set of hypotheses explaining the non-entailed semantic match w.r.t. Eq. (25) from the set Φ obtained in **Step 3**.

We will now present the implementation for Steps 2-4 of our method.

Step 2. In this step the respective description trees of the inputted concept descriptions are obtained. To do this, we first impose a standardization on inputted concept definitions.

Definition 17. *Let $\mathcal{T}$ be an $\mathcal{EL}_\bot$ TBox and let C be a concept description defined w.r.t. the signature of $\mathcal{T}$. Then, C is standardized if $C = \bigsqcap_{i=0}^{n} A_i \sqcap (\bigsqcap_{i=0}^{m} (\exists r_i.C_i))$ and for all $C_i, 0 \leqslant i \leqslant m$ in C, C_i is standardized.*

The algorithm that translates a concept description into a description tree is represented in Appendix A.6 Alg. 7. Essentially, it splits the expression into two parts: the conjunction of atomic concepts, and conjunction of role restrictions. It creates a node and adds the set of concept conjuncts as a label of that node. Next, it is iterated over each role restriction and the children of the current node are constructed. Each edge is labeled w.r.t. the role names. Following this, the concepts restricted by the roles are added next in queue and the same is done for them. This follows the inductive definition of description trees presented in Section 5. Each concept description is written in OWL Manchester Syntax.

Step 3. This step contains the main algorithm for computing subtree isomorphisms between description trees. It is the main implementation of our methodology. Here are the steps of the algorithm for computing $\rho_{\subseteq}$-subtree isomorphisms between description trees:

Step 3.1 Read inputted description trees of concepts.
Step 3.2 Initialize the set of $\rho_{\subseteq}$-subtree isomorphisms Φ.
Step 3.3 Initialize the solutions tree and its root by mapping the roots of the description trees.
Step 3.4 Add the root node of the solutions tree to the queue.
Step 3.5 Obtain the next node from the solutions tree in the queue.
Step 3.6 For each pair of mapped vertices in the current node find all children in the respective description trees and partition them w.r.t. the equivalence relation in Theorem 2.
Step 3.7 Construct the relation $\mu(v,w)$ using Eq. (17) for the pair of vertices v and w.
Step 3.8 Construct $I(v,w)$ using Eq. (18).
Step 3.9 Construct and save $\mathcal{M}(v,w)$ for the current pair of vertices using Eq. (19).
Step 3.10 If there are no more pairs of vertices to evaluate go to the next step, otherwise go to **Step 3.6**.
Step 3.11 For each $\mathcal{M}(v,w)$ find all combinations of mapped vertices and construct a new node in the solutions tree and an edge between the current node and newly constructed node.
Step 3.12 If the queue is empty continue, otherwise go to **Step 3.5**.
Step 3.13 Find all complete paths in the solutions tree and construct Φ using Eq. (22).

To illustrate how we compute $\rho_{\subseteq}$-subtree isomorphisms between descriptions trees and construct the solutions tree, consider the following example TBox related to medical imaging techniques:

$$\begin{aligned}\mathcal{T} = \{&\exists\text{produces.RadioWave} \sqcap \exists\text{produces.SoundWave} \sqsubseteq \bot, \text{SoundWave} \sqcap \text{RadioWave} \sqsubseteq \bot,\\ &\text{RadioWave} \sqsubseteq \text{Non-IonizingRadiation}, \text{IonizingRadiation} \sqcap \text{Non-IonizingRadiation} \sqsubseteq \bot\},\end{aligned} \tag{26}$$

with signature $\Sigma_{\mathcal{T}} = \langle \mathcal{N}_{\mathcal{C}}, \mathcal{N}_{\mathcal{R}} \rangle$, where:

$$\begin{aligned}N_{\mathcal{C}} &= \{\text{Imaging}, \text{Magnet}, \text{Material}, \text{RadioWave}, \text{X-Ray}, \text{X-RayComputedTomography}, \text{IonizingRadiation},\\ &\qquad \text{Non-IonizingRadiation}, \text{X-RayTube}, \text{Rotation}, \text{Part}\}\\ N_{\mathcal{R}} &= \{\text{uses}, \text{performs}, \text{produces}, \text{involves}\}\end{aligned}$$

Let **CTScan**, and **X-RayScan** be two concepts defined w.r.t the signature $\Sigma_{\mathcal{T}}$, such that:

$$\begin{aligned}\textbf{CTScan} = \text{Scan} &\sqcap \exists\text{performs.X-RayComputedTomography}\\ &\sqcap \exists\text{uses.}(\text{X-RayTube} \sqcap \exists\text{involves.Rotation}) \sqcap \text{produces.X-Ray},\end{aligned} \tag{27}$$

$$\textbf{X-RayScan} = \text{Scan} \sqcap \exists\text{performs.Imaging} \sqcap \exists\text{uses.Part} \sqcap \exists\text{produces.IonizingRadiation}. \tag{28}$$

In this working example we have an abduction problem $\langle \mathcal{T}, \text{match}_{\sqsubseteq}(\text{CTScan}, \text{X-RayScan}) \rangle$. It can be easily verified that $\mathcal{T} \not\models \text{CTScan} \sqcap \text{X-RayScan} \equiv \bot$ and $\mathcal{T} \not\models \text{CTScan} \sqsubseteq \text{X-RayScan}$.

The description trees of the concepts **CTScan** and **X-RayScan**, as well as the solutions tree are shown in Figure 3.

By definition, the roots of description trees need to be mapped in order to obtain an isomorphism. Thus, we construct the root of the solutions tree carrying the mapping $\langle v_0, w_0 \rangle$ and denote the root of the solutions tree as ϕ_0. Next, the children of the roots v_0 and w_0 are partitioned w.r.t. the edge labels, from which we get:

- $N(v_0)_{/\sim} = \{[v_1], [v_2], [v_4]\}$, and
- $N(w_0)_{/\sim} = \{[w_1], [w_2], [w_3]\}$.

This partitioning is done according to the equivalence relation defined in Theorem 2. In the example, the equivalence classes contain the following vertices in T_1:

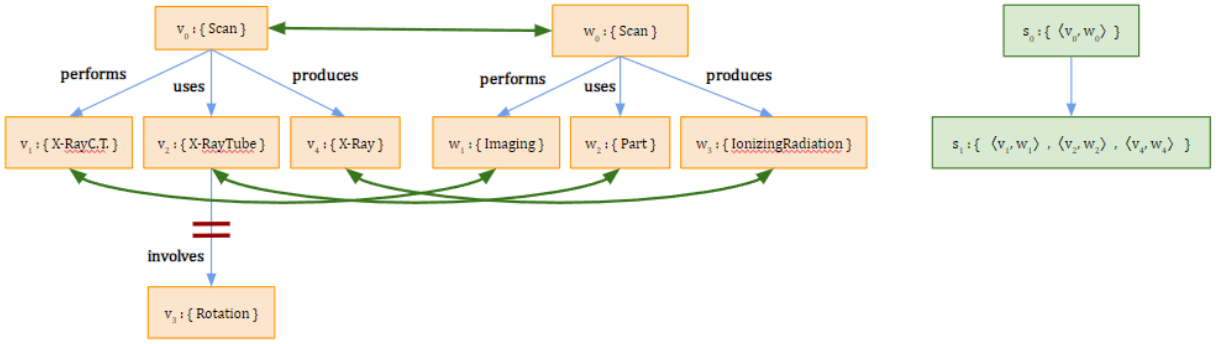

Fig. 3. A Solutions Tree containing the subtree isomorphism between the concepts CTScan and X-RayScan.

– $[v_1] = \{v_1\}, [v_2] = \{v_2\}, [v_4] = \{v_4\},$

and in T_2:

– $[w_1] = \{w_1\}, [w_2] = \{w_2\}, [w_3] = \{w_3\}.$

Further, the relation from Eq. (17) is defined for v_0 and w_0 using the sets of partitions $N(v_0)_{/\sim}$ and $N(w_0)_{/\sim}$, from which we get:

– $\mu(v_0, w_0) = \{\langle [v_1], [w_1] \rangle, \langle [v_2], [w_2], \langle [v_4], [w_3] \rangle \rangle\}.$

The relation $\mu(v_0, w_0)$ pairs the corresponding equivalence classes of the children of v_0 and w_0. An equivalence class obtained from a specific edge label in one tree is paired with an equivalence class obtained from an equal edge label in the other tree. Then, the relation $\mu(v_0, w_0)$ is used to map the corresponding equivalence classes.

We are interested in all isomorphic mappings. Thus, to find all subtree isomorphism, we need to search for injective maps between pairs of equivalence classes (Eq. (18)). For the example, the set of injective mappings between equivalence classes is:

– $I(v_0, w_0) = \{\{\langle v_1, w_1 \rangle\}, \{\langle v_2, w_2 \rangle\}, \{\langle v_4, w_3 \rangle\}\},$

Finally, all unique combinations of injections are taken using Eq. (19):

– $\mathcal{M}(v_0, w_0) = \{\{\langle v_1, w_1 \rangle, \langle v_2, w_2 \rangle, \langle v_4, w_3 \rangle\}\}.$

A new node ϕ_1 in the solutions tree is created that carries the mapped nodes from $\mathcal{M}(v_0, w_0)$, such that $\lambda_{\mathcal{V}_\Phi}(\phi_1) = \{\langle v_1, w_1 \rangle, \langle v_2, w_2 \rangle, \langle v_4, w_3 \rangle\}$. If there were more than one set of mapped vertices between the trees, then for each set a new node is constructed and added in the solutions tree. To complete this iteration, the node ϕ_1 from the solutions tree is added next in the queue. The algorithm goes over each pair of mapped vertices from the queued solutions tree node and performs the same steps. This procedure ensures that in each step only vertices that preserve the structure between the description trees will be mapped, while the ones that do not preserve the structure will not be mapped to any vertices and will induce a subtree in their respective description tree.

In the next step the mapped vertices in the pairs $\langle v_1, w_1 \rangle$, $\langle v_2, w_2 \rangle$, and $\langle v_4, w_3 \rangle$ are to be evaluated. As before, their children are partitioned and we get:

– $N(v_1)_{/\sim} = \{\}$ and $N(w_1)_{/\sim} = \{\},$
– $N(v_2)_{/\sim} = \{[v3]\}$ and $N(w_2)_{/\sim} = \{\},$ and
– $N(v_4)_{/\sim} = \{\}$ and $N(w_3)_{/\sim} = \{\},$

from which the mapping relations are formulated w.r.t. Eq. (17):

– $\mu(v_1, w_1) = \{\},$
– $\mu(v_2, w_2) = \{\},$

– $\mu(v_4, w_3) = \{\}$.

We can see that in this iteration there exist no nodes from either description tree that could be mapped in order to obtain an isomorphism. All nodes that could not be mapped in this iteration, because they are tails of edges that do not have corresponding labels and mapping them would not provide an isomorphism, are omitted from the final abducitve solution. Thus, the method stops this iteration and goes onto the next. Since no new nodes are added in the queue to be evaluated, the method returns the solutions tree with all possible $\rho_{\subseteq}$-subtree isomorphisms found in each unique path from the root to leaves in the tree.

From the solutions tree on Figure 3 we can see that the set of all complete paths is $\Pi^0 = \{\langle \phi_0, \phi_1 \rangle\}$. In this case, we only have one $\rho_{\subseteq}$-subtree isomorphism. In cases where there are more than one solution, we run a simple breadth-first search (BFS) to obtain the set of all complete paths in the solutions tree. From there, we construct the set of all weak $\rho_{\subseteq}$-subtree isomorphisms, Φ, which for our running example:

– $\Phi = \{\{\langle v_0, w_0 \rangle, \langle v_1, w_1 \rangle, \langle v_2, w_2 \rangle, \langle v_4, w_3 \rangle\}\}$,

where Φ contains the weak subtree isomorphism between the description trees T_{CTScan} and $T_{\text{X-RayScan}}$.

The isomorphisms are built from top to bottom and the equivalence relation that partitions the nodes w.r.t. the edge labels, as well as the relation $\mu(v, w)$ that partitions the equivalence classes w.r.t. corresponding edge labels in both trees, adhere to the definition of weak isomorphisms and do not allow for mapping of vertices that do not preserve the structure between description trees. Moreover, those edges and vertices that do not preserve the structure between the description trees, and consequently between the descriptions of the matching concepts, are omitted from the final set of mapped vertices, thus inducing subtrees of their respective trees. The complete algorithm for computing weak isomorphisms between $\rho_{\subseteq}$-subtrees of description trees is shown in Algorithm 1.

Step 4. The final step of the main code takes as an input a set of weak $\rho_{\subseteq}$-subtree isomorphisms between description trees. It then iterates over all weak $\rho_{\subseteq}$-subtree isomorphisms, and for each one it construct the missing relation between concepts and/or role restrictions w.r.t. Eq. (25), thus extending the set of weak isomorphisms to $\mathcal{T} \cup \mathcal{H}$-isomorphisms w.r.t. Definitions 10 and 12. This is done for each obtained weak $\rho_{\subseteq}$-subtree isomorphism between the description trees. The steps of this algorithm are as follows:

Step 4.1 Read the set of weak $\rho_{\subseteq}$-subtree isomorphisms, Φ.
Step 4.2 For each $\phi \in \Phi$, update the labels of vertices using Eq. (24).
Step 4.3 Construct the hypothesis for each ϕ w.r.t. Definitions 10 and 12.

The algorithm for constructing the hypotheses from a set of weak $\rho_{\subseteq}$-subtree isomorphisms is shown in Algorithm 6 Appendix A.5. The algorithm traverses over each weak $\rho_{\subseteq}$-subtree isomorphism and constructs the hypothesis as in Eq. (25). All vertices that are not included in the isomorphic mappings, because they do not satisfy the conditions for a weak isomorphism, induce $\rho_{\subseteq}$-subtrees of their respective description trees. The labels of the vertices that are included in the weak $\rho_{\subseteq}$-subtree isomorphism are then updated with the role restrictions of the concepts in the labels of vertices that were omitted from the weak $\rho_{\subseteq}$-subtree isomorphism, according to Definition 12.

In our working example, we have one isomorphism, $\Phi = \{\langle v_0, w_0 \rangle, \langle v_1, w_1 \rangle, \langle v_2, w_2 \rangle, \langle v_4, w_3 \rangle\}$, from which we obtain the hypothesis:

– $\mathcal{H} = \{\text{X-RayC.T.} \sqsubseteq \text{Imaging}, \text{X-RayTube} \sqcap \exists \text{involves.Rotation} \sqsubseteq \text{Part}, \text{X-Ray} \sqsubseteq \text{IonizingRadiation}\}$,

In natural language this hypothesis states that *"X-Ray computed tomography is a type of imaging."*, *"X-Ray tube that rotates is a part."* and *"X-Ray is a type of ionizing radiation."*. In reality, an X-Ray computed tomography is a type of an X-Ray scan, because it is based on the same technique. In fact, an X-Ray computed tomography is a specific type of an X-Ray scan. The hypothesis clearly states what is missing from our background knowledge in order for the concepts **X-RayC.T.** and **X-RayScan** to be in a plugin match.

Algorithm 1 *subtreeIsomorphisms*(T_C, T_D)

```
Φ = {}
ST_nodes ← {0 : [⟨v_0, w_0⟩]}
ST_edges ← {}
queue ← [0]
i_solution ← 1
while |queue| ≠ 0 do
  n ← queue.poll()
  mappingsToEvaluate ← ST_nodes.get(n)
  uniqueMappings ← []
  for all ⟨v, w⟩ ∈ mappingsToEvaluate do
    currentMappingsEQC ← getMappingsOfEquivalenceClasses(v, w, T_C, T_D)
    uniqueMappings.add(currentMappingsEQC)
  end for
  for all x ∈ getCartesianProduct(uniqueMappings) do
    newSTNode ← []
    for all y ∈ x do
      if |y| = 0 then
        continue
      end if
      newSTNode.addAll(y)
    end for
    if |newSTNode| = 0 then
      continue
    end if
    ST_nodes.put(i_solution, newSTNode)
    ST_edges.add(⟨n, i_solution⟩)
    queue.add(i_solution)
    i_solution ← i_solution + 1
  end for
end while
sol ← 1
for all π ∈ allCompletePaths(ST_nodes, ST_edges) do
  φ ← []
  for all node ∈ π do
    for all ⟨v, w⟩ ∈ ST_nodes.get(node) do
      φ.add(⟨v, w⟩)
    end for
  end for
  Φ.put(sol, φ)
  sol ← sol + 1
end for
return Φ
```

6.2. Experiments

Since there is no standardized benchmark for $\mathcal{EL}_\perp$ abduction, we constructed random description trees with arbitrary concepts and performed abduction using our method. [3].

Simulations. The experiments were conducted on a 64-bit operating system running on Windows 10, CPU Intel(R) Core(TM) i5-7400 CPU @ 3.00GHz, RAM 8.00 GB. Concept descriptions were generated randomly with different number of concepts and roles, with allowed repetition of role restrictions on various concepts. This way we also cover extreme cases in which concepts are restricted by the same role multiple times, possibly leading to a large number of abductive solutions. We then obtained the description trees for the randomly generated concept descriptions and performed abduction. We report on three scenarios. For each scenario we ran 250 simulations with the given parameters. The non-entailment of interest is $\mathcal{T} \not\models C \sqsubseteq D$, where C and D are the randomly generated concept descriptions. For each solution we constructed a new ontology, such that $\mathcal{T} = \emptyset$ in which we added the atomic concepts and roles and the matching concept definitions from those trees, thus generating a signature for the knowledge base and a non-entailment to explain. We then subsequently added the hypotheses to each distinct ontology and checked whether $\mathcal{T} \cup \mathcal{H} \models C \sqsubseteq D$, i.e. whether the non-entailment has been explained by the certain hypothesis.

We measured the mean, median, and maximal values of the cardinalities of vertex sets ($|V_1|, |V_2|$), the number of hypotheses (solutions) to the given abduction problem ($\#H$), the size of the solutions ($|H|$), the time needed to compute the subtree isomorphisms and checked whether the generated hypotheses explain the non-entailments. The simulations and results are shown in Table 1. In addition, we obtained the average branching factors of the

Table 1

Results from the experiments.

mean | median | max

	V_1	V_2	$\#\mathcal{H}$	$\lvert\mathcal{H}\rvert$	$t[s]$	$\mathcal{T} \cup \mathcal{H} \models \eta[\%]$
1	2.82 | 3.0 | 8	2.79 | 3.0 | 9	1.04 | 1.0 | 2.0	1.27 | 1.0 | 3.5	0.0 | 0.0 | 0.004	100.0
2	7.73 | 8.0 | 14	7.98 | 8.0 | 14	2.33 | 2.0 | 18.0	2.7 | 3.0 | 6.0	0.002 | 0.0 | 0.058	100.0
3	12.38 | 12.0 | 20	12.68 | 12.5 | 21	15.95 | 6.0 | 216.0	4.52 | 4.33 | 8.67	0.006 | 0.002 | 0.236	100.0

description trees and the sets of edge labels of description trees. The set of edge labels for a discription tree $T = \langle \mathcal{V}, \mathcal{E}, v_0, \lambda_\mathcal{V}, \lambda_\mathcal{E} \rangle$ is defined as $\Sigma_r(\mathcal{E}) = \{\lambda_\mathcal{E}(e) \mid e \in \mathcal{E}\}$. We then calculate the ratio between the average branching factors and the Jaccard index of the sets of edge labels defined as:

$$J(\Sigma_r(\mathcal{E}_1), \Sigma_r(\mathcal{E}_2)) = \frac{|\Sigma_r(\mathcal{E}_1) \cap \Sigma_r(\mathcal{E}_2)|}{|\Sigma_r(\mathcal{E}_1) \cup \Sigma_r(\mathcal{E}_2)|} \tag{29}$$

The results demonstrate that the method can practically and correctly compute hypotheses to explain $\mathcal{EL}_\perp$ non-entailments. For smaller concept descriptions in the first and second scenario that produce smaller trees the method computes the hypotheses for the given abduction problem almost instantaneously. Even for larger concept descriptions that produce larger trees (Scenario 3) with 20 vertices the method is quite versatile. However, there is some deviation in the results that show distinct cases where the method needs more time to compute subtree isomorphisms due to the frequency of repetition of edge labels in description trees.

Figure 4 shows the dependence between the branching factors and similarity of edge labels in description trees and the number of solutions for the specific case. This is presented for all simulation scenarios. In all cases we can generally see a positive correlation between the number of a hypotheses and the time needed to compute those. In addition, we can see that a low ratio between the average branching factors of description trees, i.e. higher difference between the number of vertices in description trees, implies a higher number of hypotheses. For the similarity of

[3]The code used to run the simulations and solve the working example is available on a GitHub repository https://github.com/Gochev-Ivan/An-Abduction-based-Method-for-Explaining-Non-entailments-of-Semantic-Matching

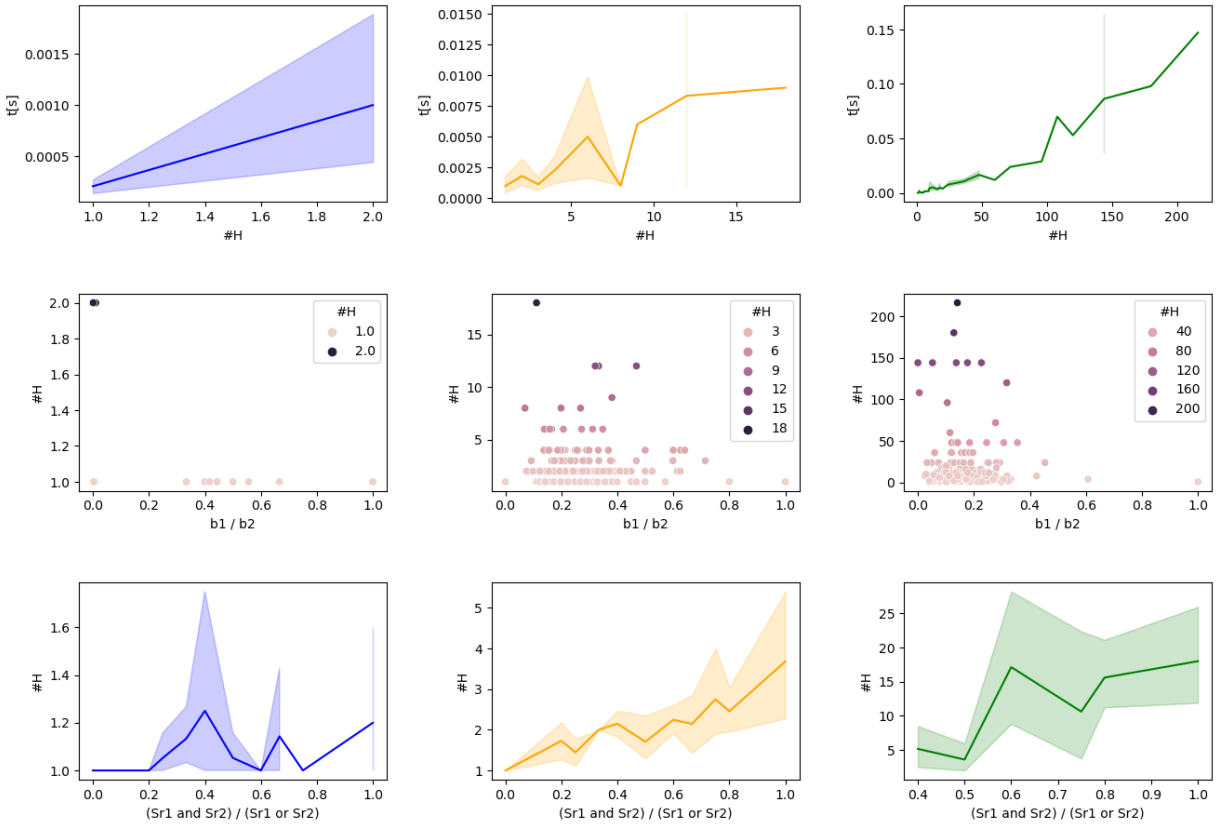

Fig. 4. Simulation results: Number of Hypotheses vs time (top), Ratio of Branching Factors vs Number of Hypotheses (middle), Similarity of Edge Labels vs Number of Hypotheses (bottom).

edge labels, we can notice from the graphs that the more similar the sets of edge labels of description trees are, the more hypotheses will be generated. This is due to our approach of partitioning the vertices on specific depths in the trees based on the labels of edges. In order to obtain isomorphic mappings we need to map edges that have same edge labels (Definition 9 condition 2). Moreover, we obtain injective mappings between vertices that belong in partitions coming from same edge labels. Consequently, if there is a lower similarity of the sets of edge labels, or even no common edge labels, then there will be a lower number (or even none) partitions and vertices to map. Thus, the method omits those mappings between vertices that do not represent an isomorphism. These are the cases in which the method almost instantaneously computes subtree isomorphisms. On the other hand, higher similarity of edge labels in the trees may lead to a large number of hypotheses. Thus, a limitation of our method is that it may not be able to practically compute all hypotheses in specific cases where the descriptions of matching concepts contain a large number of same roles, and a timeout should be introduced.

7. Discussion

Although our method is focused on explaining non-entailments of semantic matching, our main contribution is the extension of the state of the art, by generalizing abduction to axioms consisting of role restrictions too, and not only of atomic concepts.

The utilization of homomorphisms between description trees to solve abduction problems in $\mathcal{EL}$, that provide minimal connections between concepts was initially presented in [15]. The abduction consists of axioms that carry

atomic concepts only. Even though our method finds direct relations between concepts in semantic matching, instead of finding connecting concepts in some background knowledge, the abduction of role restrictions that we introduce can enhance the method in [15] by finding connecting concepts that would otherwise be excluded if the abductive process is restricted to concepts only. We achieved this by inducing specific subtrees termed $\rho_{\subseteq}$-subtrees, which retain the original root and help identify non-isomorphic parts of description trees. The final solution contained the isomorphic parts, which enabled abduction of concepts, whereas the non-isomorphic parts of trees were used to update existing labels, which enabled abduction of role restrictions. Thus, state-of-the-art methods that use similar approaches could benefit by introducing abduction of role restrictions, instead of constraining the set of abducibles to concepts only. In addition to our method preserving the structure of concept definitions it does not unnecessarily omit parts relevant to the abduction.

In [16], the problem of abduction of concepts as well as role restrictions is tackled. This is done by constraining the abductive process by a set of predefined patterns that can be abducted. Apart from specializing our method for semantic matching, we do not constrain the form of expressions included in the abducted axioms, instead, our method is constrained by the signature of the background knowledge.

In [45, 47, 54] we can see more specifically abduction being used for semantic matching in more expressive DLs. In comparison, instead of reducing matching concept descriptions or introducing new concepts to reach a positive semantic match, our method searches for direct relations between concepts and role restrictions in matching concepts, thus preserving the original definitions of matching concepts.

8. Conclusion

We presented a method for explaining non-entailments of semantic matching in $\mathcal{EL}_{\bot}$ ontologies, by computing subtree isomorphisms between graphical representations of concept descriptions, i.e. description trees. To compute isomorphic mappings, we started by partitioning the sets of neighboring nodes based on edge labels. The partitioned nodes from the respective trees were then mapped w.r.t. the labels. We formalized our approach and tested our method. The results showed that the method provides correct solutions within a reasonable time frame. We also discussed how our method could be used to enhance related methods, to achieve better results.

There are several ways to improve our method. First, the complexity of our method rises with the size of concept descriptions, therefore the size of description trees. Moreover, a combinatorial explosion occurs when there is higher repetition of roles in concept descriptions. A more thorough complexity analysis needs to be carried out. To improve the search for isomorphic mappings, we want to introduce heuristics based on the level of information concepts and roles carry in concept descriptions w.r.t. the background knowledge. This would return the hypotheses that carry the most information and exclude those that are similar or carry less information. In addition, we are researching the possibility of computing isomorphisms from paths in description trees, since paths from the root to a leaf node in rooted trees are unique, and preserved up to isomorphism, they could limit the search space even further. Finally, we want to look into the possibility of extending our method to more expressive description logics.

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Appendix A. Functions

Here are additional functions used in the main implementation of our method.

A.1. equivClasses *function*

Algorithm 2 $equivClasses(x, Nx, T)$

$Nx_{/\sim} \leftarrow \{\}$
for all $y \in Nx$ **do**
  eqClass $\leftarrow \{y\}$
  for all $z \in Nx$ **do**
    if $\xi(\langle x, y\rangle) = \xi(\langle x, z\rangle)$ **then**
      eqClass.$add(z)$
    end if
  end for
  $Nx_{/\sim}.add(eqClass)$
end for
return $Nx_{/\sim}$

A.2. μ function

Algorithm 3 $\mu(v, w, Nv_{/\sim}, Nw_{/\sim}, T_C, T_D)$

$\mu_{vw} \leftarrow \{\}$
for all $x_{equivClass} \in Nv_{/\sim}$ **do**
  for all $y_{equivClass} \in Nw_{/\sim}$ **do**
    $x \leftarrow pickRepresentative(x_{equivClass})$
    $y \leftarrow pickRepresentative(y_{equivClass})$
    if $\xi_C(\langle v, x\rangle) = \xi_D(\langle w, y\rangle)$ **then**
      $\mu_{vw}.add(\langle x_{equivClass}, y_{equivClass}\rangle)$
    end if
  end for
end for
return μ_{vw}

A.3. getMappingsOfEquivalenceClasses *function*

Algorithm 4 $getMappingsOfEquivalenceClasses(v, w, T_C, T_D)$

```
mappingsEQC ← []
Nv ← {x | x ∈ V_C and ⟨v, x⟩ ∈ E_C}
Nw ← {x | x ∈ V_D and ⟨v, x⟩ ∈ E_D}
Nv_/∼ ← equivClasses(v, Nv, T_C)
Nw_/∼ ← equivClasses(w, Nw, T_D)
μ_vw ← μ(T_C, T_D, v, w, Nv_/∼, Nw_/∼)
M_i ← []
for all r ∈ μ_vw do
    M_i.add(mapEquivalenceClasses(r))
end for
for all a ∈ getCartesianProduct(M_i) do
    t ← []
    for all b ∈ a do
        t.addAll(b)
    end for
    mappingsEQC.add(t)
end for
return mappingsEQC
```

A.4. mapEquivalenceClasses *function*

Algorithm 5 $mapEquivalenceClasses(\langle x_{equivClass}, y_{equivClass}\rangle$

```
injectiveMap ← {}
if |x_equivClass| ⩽ |y_equivClass| then
    for all p ∈ permutations(y_equivClass, |x_equivClass|) do
        combPair ← []
        for i ← 0; i < |p|; i++ do
            combPair.add(⟨x_equivClass.get(i), p.get(i)⟩)
        end for
        injectiveMap.add(combPair)
    end for
else
    for all p ∈ permutations(x_equivClass, |y_equivClass|) do
        combPair ← []
        for i ← 0; i < |p|; i++ do
            combPair.add(⟨p.get(i), y_equivClass.get(i)⟩)
        end for
        injectiveMap.add(combPair)
    end for
end if
return injectiveMap
```

A.5. constructHypotheses *function*

Algorithm 6 *constructHypotheses*$(\Phi, \sqsubseteq\sqsupseteq, T_C, T_D)$

$\mathcal{H} \leftarrow \{\}$
for all $\phi \in \Phi$ **do**
  for all $x \in \mathcal{V}_C \wedge x \notin \phi$ **do**
    $\lambda_{\mathcal{V}_C}(v) \leftarrow \bigsqcap \lambda_{\mathcal{V}_C}(v) \sqcap \bigsqcap_{\langle v,x\rangle \in \mathcal{E}_C} \exists \lambda_{\mathcal{E}_C}(\langle v,x\rangle).C_{T_C(v)}$
  end for
  for all $y \in \mathcal{V}_D \wedge y \notin \phi$ **do**
    $\lambda_{\mathcal{V}_D}(w) \leftarrow \bigsqcap \lambda_{\mathcal{V}_D}(w) \sqcap \bigsqcap_{\langle w,y\rangle \in \mathcal{E}_D} \exists \lambda_{\mathcal{E}_D}(\langle w,y\rangle).C_{T_D(w)}$
  end for
  $h \leftarrow \{\}$
  for all $\langle v,w\rangle \in \phi$ **do**
    $h.add(\bigsqcap \lambda_{\mathcal{V}_C}(v) \sqsubseteq\sqsupseteq \bigsqcap \lambda_{\mathcal{V}_D}(w))$
  end for
  $\mathcal{H}.add(h)$
end for
return $\mathcal{H}$

A.6. getDescriptionTree

Algorithm 7 $getDescriptionTree(C)$

Require: C in normal form

```
i_x ← 0
i_y ← 1
queue ← [⟨C, i_x⟩]
while |queue| ≠ 0 do
    ⟨C_i, i_x⟩ ← queue.poll()
    conjuncts ← getExpressionParts(getConjuncts(C_i))
    for all conjunct ∈ conjuncts do
        edgeLabel ← ""
        conceptLabels ← ""
        if |conjunct| = 1 then
            arr ← conjunct.get(0).split(" some ")
            edgeLabel ← arr.get(0)
            conceptLabels ← arr.get(1)
        else
            for x ← 0; x < |conjunct.get(0)|; x++ do
                if conjunct.get(0).charAt(x) = " " then
                    break
                end if
                edgeLabel ← edgeLabel + conjunct.get(0).charAt(x)
            end for
            for x ← 0; x < |conjunct.get(1)|; x++ do
                if conjunct.get(1).charAt(x) = "(" then
                    break
                end if
                conceptLabels ← conceptLabels + conjunct.get(1).charAt(x)
            end for
        end if
        conceptArr ← conceptLabels.split(" and ")
        l_1 ← getTopLevelConjuncts(C_i)
        l_2 ← []
        for x ← 0; x < |conceptArr|; x++ do
            l_2.add(conceptArr.get(x))
        end for
        T.add(i_x, i_y, edgeLabel, l_1, l_2)
        if |conjunct| ≠ 1 then
            if |conjunct.get(1)| ≠ 0 then
                queue.add(⟨conjunct.get(1), i_y⟩)
            end if
        end if
        i_y ← i_y + 1
    end for
end while
return T
```