Ontology-mediated query answering over temporal and inconsistent data

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Abstract. Stream-based reasoning systems process data stemming from different sources and that are received over time. In this kind of application, reasoning needs to cope with the temporal dimension and should be resilient against inconsistencies in the data. Motivated by such settings, this paper addresses the problem of handling inconsistent data in a temporal version of ontology-mediated query answering. We consider a recently proposed temporal query language that combines conjunctive queries with operators of propositional linear temporal logic and extend to this setting three inconsistency-tolerant semantics that have been introduced for querying inconsistent description logic knowledge bases. We investigate their complexity for $\mathcal{EL}_\bot$ and DL-Lite$_\mathcal{R}$ temporal knowledge bases. In particular, we consider two different cases, depending on the presence of negations in the query. Furthermore we complete the complexity picture for the consistent case. We also provide two approaches toward practical algorithms for inconsistency-tolerant temporal query answering.

Keywords: temporal query answering, inconsistency-tolerance, description logics, computational complexity

1. Introduction

For applications that rely on sensor data, such as context-aware applications, ontologies can enrich and abstract the (numerical) stream data by means of background knowledge. This richer view on the data often results in more query results than over the data alone. Since the collected data usually provides an incomplete description of the observed system, the closed world assumption employed by database systems, where facts not present are assumed to be false, is not appropriate. Most applications that rely on sensor streams observe some kind of running system over time. In order to be able to react to the behaviour of the observed system, they need to employ some representation of temporal information and a query mechanism that can reference this temporal information. If the sources of the collected data are not reliable, as it might be in case of faulty sensors, the internal representation of the observations may contain inconsistencies. In such cases, query mechanisms that rely on logical reasoning are effectively useless, as everything would follow from an inconsistent knowledge base. As a counter measure to this effect, inconsistency-tolerant semantics for answering ontology-mediated queries have been devised. In this paper, we investigate combinations of inconsistency-tolerant and temporal query answering, which addresses two aspects vital to stream reasoning and complex event processing.

In many stream reasoning systems, the collected data is transformed into an abstract logical representation, and situation recognition is performed by some kind of logical inference over the abstract logical representation. There are stream reasoning approaches based on rules, such as answer set programming [1–3], (datalog) rules and approaches based on ontology languages [4–7]. While the former apply closed world semantics, the later work under the open world semantics and thereby can handle incomplete information gracefully. The ontology-based approaches mostly employ the framework of *ontology-mediated queries*, where

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forms of conjunctive queries are answered over data that is enriched by an ontology, to perform situation recognition. The ontology languages that are investigated for situation recognition are mostly those where reasoning is of lower computational complexity in order to obtain systems with low execution times.

In this paper, we investigate the lightweight description logics (DLs) $\mathcal{EL}_\bot$ and DL-Lite$_\mathcal{R}$, for which answering conjunctive queries is tractable (respectively in P and AC^0 w.r.t. the size of the data). The low complexity for query answering in DL-Lite$_\mathcal{R}$ made it the choice for the OWL 2 QL profile [8] in the latest version of OWL [9], the W3C-standardized ontology language for the Semantic Web. For similar reasons, the logic $\mathcal{EL}$ was picked as the core of the OWL 2 EL profile. Both DL-Lite$_\mathcal{R}$ and $\mathcal{EL}_\bot$ admit to use database systems to answer conjunctive queries and are thus good candidates for implementing ontology-based stream reasoning. In DL-Lite$_\mathcal{R}$, the query can be rewritten using the information from the ontology such that the resulting query can be evaluated over a finite interpretation, i.e. a database [10]. For query answering in $\mathcal{EL}$, the data is augmented in a query-independent way to build a canonical interpretation, then the query is evaluated over this model and unsound answers are filtered out [11].

In stream reasoning approaches in general, the temporal information is often represented by associating data with the time point at which it was collected. Regarding the language in which queries can be formulated, many variations that capture the temporal aspect have been studied in recent research [2–4, 12]. Window based approaches admit to concentrate on recent substreams when answering queries over the data, and are the most prominent in implemented systems [2–4]. Ontology-based approaches mostly cover classical temporal logics such as linear temporal logic (LTL) [13] (see [5, 14–17]) or metric variants of temporal logics [18, 19] to enrich the query language. For a recent overview on temporal ontology-mediated querying see [19, 20]. Ontology-based approaches for stream reasoning often admit the use of temporal operators only in the query language and use classical ontologies without any temporal operators together with sequences of datasets. Each dataset in such a sequence contains data collected at the same time point. The ontology together with the sequence of datasets constitute the *temporal knowledge base*. Queries can then refer to the different time points by means of temporal operators. This kind of setting has been intensively investigated for *temporal conjunctive queries*, that is, queries with temporal operators from LTL appearing in front of Boolean combinations of atoms, for expressive DLs in [5, 21], for DL-Lite$_\mathcal{R}$ in [6], and for $\mathcal{EL}$ in [22]. We base our study in this paper on this general setting.

For stream reasoning systems, erroneous data sources can be a severe problem, as for instance pointed out in [23]. If inconsistencies arise in the knowledge base, the logical reasoning mechanisms are rendered useless. There are several directions of research to cope with this problem. While some employ non-monotonic reasoning techniques [24, 25], others try to resolve the inconsistencies [26] directly or perform reasoning with respect to inconsistency-tolerant semantics (see [27] for a recent overview). We follow the latter road in this paper, since the techniques developed there are tailored to ontology-mediated queries and often of lower complexity than the other approaches for resolving inconsistencies. A prominent approach for inconsistency-tolerant reasoning is to consider repairs of the knowledge base, i.e., maximal consistent subsets of the data, and then to perform query answering with respect to these subsets. Arguably the most natural and well-known inconsistency-tolerant semantics is the *AR semantics* [28, 29], inspired by consistent query answering in the database setting [30], which considers the queries that hold in *every repair*. However, AR query answering is intractable even for very simple ontologies [31], which leads [28, 29] to propose an approximation of AR tractable for DL-Lite$_\mathcal{R}$, namely the *IAR semantics*, which queries the *intersection of the repairs*. Beside its better computational properties, this semantics is more cautious, since it provides answers supported by facts that are not involved in any contradictions, so it may be interesting in our setting when the observed system should change its behaviour only if some situation has been recognised with a very high confidence. Finally, the *brave semantics* [32] returns every answer that holds in *some repair*, so is supported by some consistent set of facts. This less cautious semantics may be relevant for context recognition, when critical situations must imperatively be handled.

For the two DLs to be investigated in this paper, answering of (atemporal) conjunctive queries under these inconsistency-tolerant semantics has already been investigated for DL-Lite$_\mathcal{R}$ in [28, 29, 32, 33] and for $\mathcal{EL}_\bot$ in [27, 34]. Attention has then turned to the problem of designing algorithms and implementing these alternative semantics. Most work has focused on the IAR semantics and dialects of DL-Lite, due to the aforementioned tractability result [29, 35, 36]. A no-table exception is the CQAPri system, which implements all three mentioned semantics—AR, IAR and brave—for DL-Lite$_{\mathcal{R}}$ knowledge bases [37, 38].

So far, inconsistency-tolerant semantics have not been investigated in combination with temporal reasoning. In this paper, we lift inconsistency-tolerant semantics to the case of answering temporal conjunctive queries over lightweight DL temporal knowledge bases.

1.1. Our contributions

This article extends the conference paper [39] on temporal query answering in DL-Lite$_{\mathcal{R}}$ over inconsistent data, where the complexity of answering queries with LTL operators, but without negations, over DL-Lite$_{\mathcal{R}}$ temporal knowledge bases was investigated. The considered ontologies admit the use of rigid predicates, which are predicates that do not change their interpretation over time. The initial results were obtained for the three inconsistency-tolerant semantics AR, IAR and brave and with respect of three cases of rigid predicates: no rigid predicates, rigid concepts only, and rigid concepts together with rigid roles.

Compared to the conference version, the present article includes new complexity results (all results for $\mathcal{EL}_\bot$, as well as some results for DL-Lite$_{\mathcal{R}}$). It also extends the set of temporal operators, distinguishing bounded and unbounded variants of the future LTL operators, in order to cover the two different settings that have been investigated for temporal query answering in the literature, where temporal knowledge bases are interpreted w.r.t. finite or infinite sequences of interpretations. Furthermore, we investigate both the case where negation is admitted in the query language and the case where it is not.

The complexity upper bounds are often obtained by non-deterministic procedures that require for instance to guess repairs, which may not be feasibly computed in practice. Thus algorithms that lend themselves to implementation are still to be devised. We make two contributions toward practical algorithms for temporal inconsistency-tolerant query answering. The first is a polynomial reduction of reasoning in the presence of rigid predicates to reasoning without such predicates by propagating the rigid facts in the sequence of datasets. The second is to identify cases where in the absence of rigid predicates the well-known methods for classical temporal query answering and (atemporal) inconsistency-tolerant query answering can straightforwardly be combined. We show that for the IAR semantics, this yields a sound and complete algorithm. For the AR semantics, such a combination of the algorithms always yields a sound approximation, and additionally yields a sound and complete procedure if the query contains only a restricted set of operators.

This paper is structured as follows. In the next section, we introduce the basic notions of DLs, query answering, inconsistency-tolerant semantics for atemporal knowledge bases and the temporal setting. We also discuss earlier complexity results. In Section 3, we lift the introduced inconsistency-tolerant semantics to temporal query answering over inconsistent data. Section 4 gives an overview over the achieved complexity results. General versions of algorithms for testing (non-)entailment of temporal conjunctive queries under the different semantics are described in Section 5 in preparation of the complexity analysis. Section 6 shows data and combined complexity of inconsistency-tolerant temporal query answering for DL-Lite$_{\mathcal{R}}$ and $\mathcal{EL}_\bot$ for the case where the query language admits negation. In Section 7, we complete the complexity picture of temporal query answering under classical semantics by investigating the case where the query does not contain negation. We then built on these results to provide the complexity of inconsistency-tolerant temporal query answering for queries without negations in Section 8. Finally, Section 9 investigates two approaches for practical implementations that allow to employ well-known methods. The article ends with a section on conclusions and future work.

To improve readability, some of the proofs have been moved to the appendix and are only sketched in the main text.

2. Preliminaries

We briefly recall the syntax and semantics of DLs and the three inconsistency-tolerant semantics we consider, and then introduce the setting of temporal query answering we use.

Syntax. A DL *knowledge base (KB)* $\mathcal{K}$ consists of an ABox $\mathcal{A}$ and a TBox $\mathcal{T}$, both constructed from three countably infinite sets: a set $\mathsf{N_C}$ of *concept names* (unary predicates), a set $\mathsf{N_R}$ of *role names* (binary predicates), and a set $\mathsf{N_I}$ of *individual names* (constants). The *ABox* (dataset) is a finite set of *concept assertions* $A(a)$ and *role assertions* $R(a,b)$, where $A \in \mathsf{N_C}$,

$R \in \mathsf{N_R}$, $a, b \in \mathsf{N_I}$. The *TBox* (ontology) is a finite set of axioms whose form depends on the particular DL.

In DL-Lite$_\mathcal{R}$, TBox axioms are either *concept inclusions* $B \sqsubseteq C$ or *role inclusions* $P \sqsubseteq S$, built according to the following syntax, where $A \in \mathsf{N_C}$ and $R \in \mathsf{N_R}$:

$$B := A \mid \exists P \quad C := B \mid \neg B$$

$$P := R \mid R^- \quad S := P \mid \neg P.$$

Inclusions of the form $B_1 \sqsubseteq B_2$ or $P_1 \sqsubseteq P_2$ are called *positive inclusions* (PI), those of the form $B_1 \sqsubseteq \neg B_2$ or $P_1 \sqsubseteq \neg P_2$ are called *negative inclusions* (NI).

In $\mathcal{EL}_\bot$, the TBox contains concept inclusions of the form $C_1 \sqsubseteq C_2$, where C_1 and C_2 are built according to the following syntax, where $A \in \mathsf{N_C}$ and $R \in \mathsf{N_R}$:

$$C := \top \mid \bot \mid A \mid \exists R.C \mid C \sqcap C.$$

An $\mathcal{EL}_\bot$ inclusion of the form $C_1 \sqcap C_2 \sqsubseteq \bot$ can also be written in the form of a negative inclusion $C_1 \sqsubseteq \neg C_2$.

Semantics. An *interpretation* is a tuple of the form $\mathcal{I} = \langle \Delta^\mathcal{I}, \cdot^\mathcal{I} \rangle$, where $\Delta^\mathcal{I}$ is a non-empty set and $\cdot^\mathcal{I}$ maps each $a \in \mathsf{N_I}$ to $a^\mathcal{I} \in \Delta^\mathcal{I}$, each $A \in \mathsf{N_C}$ to $A^\mathcal{I} \subseteq \Delta^\mathcal{I}$, and each $R \in \mathsf{N_R}$ to $R^\mathcal{I} \subseteq \Delta^\mathcal{I} \times \Delta^\mathcal{I}$. We adopt the unique name assumption, i.e., for all $a, b \in \mathsf{N_I}$, we require $a^\mathcal{I} \neq b^\mathcal{I}$ if $a \neq b$. The function $\cdot^\mathcal{I}$ is straightforwardly extended to general concepts and roles, e.g., $\top^\mathcal{I} = \Delta^\mathcal{I}$, $\bot^\mathcal{I} = \emptyset$, $(R^-)^\mathcal{I} = \{(d, e) \mid (e, d) \in R^\mathcal{I}\}$, $(\neg R)^\mathcal{I} = \Delta^\mathcal{I} \times \Delta^\mathcal{I} \setminus R^\mathcal{I}$, $(\exists P)^\mathcal{I} = \{d \mid \exists e : (d, e) \in P^\mathcal{I}\}$, $(\exists P.C)^\mathcal{I} = \{d \mid \exists e : (d, e) \in P^\mathcal{I}, e \in C^\mathcal{I}\}$ $(C_1 \sqcap C_2)^\mathcal{I} = C_1^\mathcal{I} \cap C_2^\mathcal{I}$.

An interpretation $\mathcal{I}$ *satisfies* an inclusion $G \sqsubseteq H$ if $G^\mathcal{I} \subseteq H^\mathcal{I}$; it satisfies $A(a)$ (resp. $R(a, b)$) if $a^\mathcal{I} \in A^\mathcal{I}$ (resp. $(a^\mathcal{I}, b^\mathcal{I}) \in R^\mathcal{I}$). We call $\mathcal{I}$ a *model* of $\mathcal{K} = \langle \mathcal{T}, \mathcal{A} \rangle$ if $\mathcal{I}$ satisfies all axioms in $\mathcal{T}$ and all assertions in $\mathcal{A}$. A KB is *consistent* if it has a model, and we say that an ABox $\mathcal{A}$ is $\mathcal{T}$*-consistent* (or simply *consistent* if $\mathcal{T}$ is clear from the context), if the KB $\langle \mathcal{T}, \mathcal{A} \rangle$ is consistent.

Queries. A *conjunctive query* (CQ) takes the form $q = \exists \vec{y}.\psi(\vec{x}, \vec{y})$, where ψ is a conjunction of atoms of the form $A(t)$ or $R(t, t')$, with t, t' individual names or variables from $\vec{x} \cup \vec{y}$. We call the variables in $\vec{x}$ the *free variables in q*. A CQ is called *Boolean* (BCQ) if it has no free variables (i.e., $\vec{x} = \emptyset$). A BCQ q is satisfied by an interpretation $\mathcal{I} = \langle \Delta^\mathcal{I}, \cdot^\mathcal{I} \rangle$, written $\mathcal{I} \models q$, if there is a homomorphism π mapping the variables and individual names of q into $\Delta^\mathcal{I}$ such that: $\pi(a) = a^\mathcal{I}$ for every $a \in \mathsf{N_I}$, $\pi(t) \in A^\mathcal{I}$ for every concept atom $A(t)$ in ψ, and $(\pi(t), \pi(t')) \in R^\mathcal{I}$ for every role atom $R(t, t')$ in ψ. A BCQ q is *entailed* from $\mathcal{K}$, written $\mathcal{K} \models q$, iff q is satisfied by every model of $\mathcal{K}$. Given a CQ q with free variables $\vec{x} = (x_1, \dots, x_k)$ and a tuple of individuals $\vec{a} = (a_1, \dots, a_k)$, $\vec{a}$ is a *certain answer* to q over $\mathcal{K}$ if $\mathcal{K} \models q(\vec{a})$, where $q(\vec{a})$ is the BCQ resulting from replacing each x_j in q by a_j.

Inconsistency-tolerant semantics. A *repair* of $\mathcal{K} = \langle \mathcal{T}, \mathcal{A} \rangle$ is an inclusion-maximal subset of $\mathcal{A}$ that is $\mathcal{T}$-consistent. We consider three semantics based on repairs that have been previously introduced in the literature [28, 29, 32]. A tuple $\vec{a}$ is an answer to q over $\mathcal{K}$ under

- *AR semantics*, written $\mathcal{K} \models_{\text{AR}} q(\vec{a})$, iff $\langle \mathcal{T}, \mathcal{A}' \rangle \models q(\vec{a})$ for *every repair* $\mathcal{A}'$ of $\mathcal{K}$;
- *IAR semantics*, written $\mathcal{K} \models_{\text{IAR}} q(\vec{a})$, iff $\langle \mathcal{T}, \mathcal{A}^\cap \rangle \models q(\vec{a})$ where $\mathcal{A}^\cap$ is the *intersection of all repairs* of $\mathcal{K}$;
- *brave semantics*, written $\mathcal{K} \models_{\text{brave}} q(\vec{a})$, iff $\langle \mathcal{T}, \mathcal{A}' \rangle \models q(\vec{a})$ for *some repair* $\mathcal{A}'$ of $\mathcal{K}$.

Figure 1 summarizes the complexity of BCQ entailment under the different semantics for DL-Lite$_\mathcal{R}$ and $\mathcal{EL}_\bot$. *Data complexity* is measured in the size of the ABox only, while *combined complexity* is measured in the size of the whole KB and the query. When complexity is measured w.r.t. the size of the KB (ABox and TBox), it is called *KB complexity*. For DL-Lite$_\mathcal{R}$ and $\mathcal{EL}_\bot$, CQ answering under the classical semantics is in P w.r.t. KB complexity. We refer to Section 4 for a reminder on the definitions of the different complexity classes that appear in this work.

Temporal query answering. We now present our temporal framework inspired from [5] and [17].

Definition 2.1 (TKB)**.** A *temporal knowledge base* (TKB) $\mathcal{K} = \langle \mathcal{T}, (\mathcal{A}_i)_{0 \leqslant i \leqslant n} \rangle$ consists of a TBox $\mathcal{T}$ and a finite sequence of ABoxes $(\mathcal{A}_i)_{0 \leqslant i \leqslant n}$. An infinite sequence $\mathcal{J} = (\mathcal{I}_i)_{i \geqslant 0}$ of interpretations $\mathcal{I}_i = \langle \Delta, \cdot^{\mathcal{I}_i} \rangle$ over a fixed non-empty domain Δ (constant domain assumption) is a *model* of $\mathcal{K}$ iff for every $i \in [\![0, n]\!]$, $\mathcal{I}_i$ is a model of $\langle \mathcal{T}, \mathcal{A}_i \rangle$, for every $i > n$, $\mathcal{I}_i$ is a model of $\mathcal{T}$, and for every $a \in \mathsf{N_I}$ and all $i, j \geqslant 0$, $a^{\mathcal{I}_i} = a^{\mathcal{I}_j}$ (*rigidity* of individual names). *Rigid predicates* are elements from the set $\mathsf{N_{RC}} \subseteq \mathsf{N_C}$ of *rigid concepts* and the set $\mathsf{N_{RR}} \subseteq \mathsf{N_R}$ of *rigid roles*. A sequence of interpretations $\mathcal{J} = (\mathcal{I}_i)_{i \geqslant 0}$ *respects rigid predicates* iff for every $X \in \mathsf{N_{RC}} \cup \mathsf{N_{RR}}$ and all $i, j \geqslant 0$, $X^{\mathcal{I}_i} = X^{\mathcal{I}_j}$. A TKB is *consistent* if it has a model that respects rigid predicates. A sequence of ABoxes

Semantics	Data complexity		Combined complexity	
	DL-Lite$_{\mathcal{R}}$	$\mathcal{EL}_\bot$	DL-Lite$_{\mathcal{R}}$	$\mathcal{EL}_\bot$
classical	in AC^0	P-complete	NP-complete	NP-complete
AR	coNP-complete	coNP-complete	Π_2^p-complete	Π_2^p-complete
IAR	in AC^0	coNP-complete	NP-complete	$\Delta_2^p[O(\log n)]$-complete
brave	in AC^0	NP-complete	NP-complete	NP-complete

Figure 1. Complexity of BCQ entailment in DL-Lite$_{\mathcal{R}}$ [28, 32] and $\mathcal{EL}_\bot$ [27, 34]

$(\mathcal{A}_i)_{0\leqslant i\leqslant n}$ is *$\mathcal{T}$-consistent*, or simply consistent, if the TKB $\langle\mathcal{T}, (\mathcal{A}_i)_{0\leqslant i\leqslant n}\rangle$ is consistent.

It is sometimes convenient to represent a sequence of ABoxes as a set of assertions associated with timestamps, which we call *timed assertions*: $(\mathcal{A}_i)_{0\leqslant i\leqslant n}$ then becomes $\{(\alpha, i) \mid \alpha \in \mathcal{A}_i, 0 \leqslant i \leqslant n\}$.

A *rigid assertion* is of the form $A(a)$ with $A \in \mathsf{N_{RC}}$ or $R(a,b)$ with $R \in \mathsf{N_{RR}}$. We distinguish three cases depending on which predicates can be rigid: in the first case there are no rigid predicates ($\mathsf{N_{RC}} = \emptyset$ and $\mathsf{N_{RR}} = \emptyset$), in the second case there are only rigid concepts ($\mathsf{N_{RC}} \neq \emptyset$ and $\mathsf{N_{RR}} = \emptyset$), and in the last case there are both, rigid concepts and rigid roles ($\mathsf{N_{RC}} \neq \emptyset$ and $\mathsf{N_{RR}} \neq \emptyset$). Because rigid concepts can be simulated with rigid roles using a pair of concept inclusions of the form $A \sqsubseteq \exists R$, $\exists R \sqsubseteq A$, these three cases cover all interesting combinations.

We denote by $\mathsf{N}_\mathsf{C}^\mathcal{K}$, $\mathsf{N}_\mathsf{R}^\mathcal{K}$, $\mathsf{N}_\mathsf{RC}^\mathcal{K}$, $\mathsf{N}_\mathsf{RR}^\mathcal{K}$, and $\mathsf{N}_\mathsf{I}^\mathcal{K}$ respectively the sets of concepts, roles, rigid concepts, rigid roles, and individuals that occur in the TKB $\mathcal{K}$.

Definition 2.2 (TCQ)**.** *Temporal conjunctive queries* (TCQs) are built from CQs as follows: each CQ is a TCQ, and if ϕ_1 and ϕ_2 are TCQs, then so are $\neg\phi_1$ (negation), $\phi_1 \wedge \phi_2$ (conjunction), $\phi_1 \vee \phi_2$ (disjunction), $\bigcirc\phi_1$ (next), $\bullet^b \phi_1$ (bounded next), $\bigcirc^-\phi_1$ (strong previous), $\bullet^-\phi_1$ (weak previous), $\Box\phi_1$ (always), $\Box^b\phi_1$ (bounded always), $\Box^-\phi_1$ (always in the past), $\Diamond\phi_1$ (eventually), $\Diamond^b\phi_1$ (bounded eventually),$\Diamond^-\phi_1$ (some time in the past), $\phi_1 \mathsf{U}\, \phi_2$ (until), $\phi_1 \mathsf{U}^b\, \phi_2$ (bounded until), and $\phi_1 \mathsf{S}\, \phi_2$ (since). We further may use $\psi_1 \rightarrow \psi_2$ as a shortcut for $\neg\psi_1 \vee \psi_2$.

We impose the constraint that the past operators $\bigcirc^-$, $\Box^-$, $\Diamond^-$ and S cannot be nested under the unbounded future operators $\bigcirc$, $\Box$, $\Diamond$ and U (in second position).

Given a TCQ ϕ, we refer to the TCQs that occur in ϕ as *subformulas of ϕ*.

Remark 2.3 (Choice of operators)**.** The additional LTL operators W (weak until), W^- (weak since), R (release), and R^- (past release) can be expressed w.r.t. our operator basis as follows: $\phi_1 \mathsf{W}\, \phi_2 \equiv (\phi_1 \mathsf{U}\, \phi_2) \vee (\Box\phi_1)$, $\phi_1 \mathsf{W}^- \phi_2 \equiv (\phi_1 \mathsf{S}\, \phi_2) \vee (\Box^-\phi_1)$, $\phi_1 \mathsf{R}\, \phi_2 \equiv \phi_2 \mathsf{W}\, (\phi_2 \wedge \phi_1)$, and $\phi_1 \mathsf{R}^- \phi_2 \equiv \phi_2 \mathsf{W}^- (\phi_2 \wedge \phi_1)$.

We will consider in Sections 7, 8 and 9 a special setting where TCQs do not contain negation symbols, which sometimes leads to a lower computational complexity. For this reason, we did not introduce $\Box\phi$ as a shortcut for $\neg\Diamond\neg\phi$, as it is often done in the literature, but instead treat the operators $\Box$ and $\Box^b$ as native members of our query language. Similarly, since the top and bottom concepts $\top$ and $\bot$ are not allowed in every DL, $\Diamond$ (resp. $\Diamond^b$) cannot be defined using U (resp. U^b) as usual in LTL ($\Diamond\phi_1 \equiv \mathsf{true}\, \mathsf{U}\, \phi_1$), unless we allow for negation (where we can express true using $\exists x.A(x) \vee \neg\exists x.A(x)$). We thus keep all these operators in the set we consider.

Note also that since disjunctions are allowed, TCQs could be defined with unions of conjunctive queries (UCQs) instead of CQs. We use CQs for simplicity.

Definition 2.4 (TCQ answering)**.** Given a TCQ ϕ with free variables $\vec{x} = (x_1, \ldots, x_k)$ and a tuple of individuals $\vec{a} = (a_1, \ldots, a_k)$, $\phi(\vec{a})$ denotes the Boolean TCQ (BTCQ) resulting from replacing each x_j by a_j. A tuple $\vec{a}$ is an answer to ϕ in a sequence of interpretations $\mathcal{J} = (\mathcal{I}_i)_{i\geqslant 0}$ at time point p iff $\mathcal{J}, p \models \phi(\vec{a})$, where the satisfaction of a BTCQ ϕ by a sequence of interpretations $\mathcal{J}$ is defined by induction on its structure as shown in Table 1. A tuple $\vec{a}$ is a *certain answer* to ϕ over $\mathcal{K}$ at time point p, written $\mathcal{K}, p \models \phi(\vec{a})$, iff $\mathcal{J}, p \models \phi(\vec{a})$ for every model $\mathcal{J}$ of $\mathcal{K}$ that respects rigid predicates.

In addition to the standard LTL past and unbounded future operators, we introduce four bounded future operators that mimic the semantics based on finite sequences of interpretations used in [17] and similar to that of LTL on finite traces (see e.g., [40]). Indeed, while the standard way of interpreting TKBs is based on infinite sequences of interpretations, it can be relevant to limit the scope of querying to the known time points, especially in the context of data streams. For instance, a user may want to ask whether a server has been running some process since it started

Table 1
Satisfaction of BTCQs by a sequence of interpretations

ϕ	$\mathcal{J}, p \models \phi$ iff
$\exists \vec{y}.\psi(\vec{y})$	$\mathcal{I}_p \models \exists \vec{y}.\psi(\vec{y})$
$\neg\phi_1$	$\mathcal{J}, p \not\models \phi_1$
$\phi_1 \wedge \phi_2$	$\mathcal{J}, p \models \phi_1$ and $\mathcal{J}, p \models \phi_2$
$\phi_1 \vee \phi_2$	$\mathcal{J}, p \models \phi_1$ or $\mathcal{J}, p \models \phi_2$
$\bigcirc\phi_1$	$\mathcal{J}, p+1 \models \phi_1$
$\bullet^b\phi_1$	$p < n$ implies $\mathcal{J}, p+1 \models \phi_1$
$\bigcirc^-\phi_1$	$p > 0$ and $\mathcal{J}, p-1 \models \phi_1$
$\bullet^-\phi_1$	$p > 0$ implies $\mathcal{J}, p-1 \models \phi_1$
$\Box\phi_1$	$\forall k, k \geqslant p, \mathcal{J}, k \models \phi_1$
$\Box^b\phi_1$	$\forall k, p \leqslant k \leqslant n, \mathcal{J}, k \models \phi_1$
$\Box^-\phi_1$	$\forall k, 0 \leqslant k \leqslant p, \mathcal{J}, k \models \phi_1$
$\Diamond\phi_1$	$\exists k, k \geqslant p, \mathcal{J}, k \models \phi_1$
$\Diamond^b\phi_1$	$\exists k, p \leqslant k \leqslant n, \mathcal{J}, k \models \phi_1$
$\Diamond^-\phi_1$	$\exists k, 0 \leqslant k \leqslant p, \mathcal{J}, k \models \phi_1$
$\phi_1 \mathsf{U} \phi_2$	$\exists k, k \geqslant p, \mathcal{J}, k \models \phi_2$ and $\forall j, p \leqslant j < k, \mathcal{J}, j \models \phi_1$
$\phi_1 \mathsf{U}^b \phi_2$	$\exists k, p \leqslant k \leqslant n, \mathcal{J}, k \models \phi_2$ and $\forall j, p \leqslant j < k, \mathcal{J}, j \models \phi_1$
$\phi_1 \mathsf{S} \phi_2$	$\exists k, 0 \leqslant k \leqslant p, \mathcal{J}, k \models \phi_2$ and $\forall j, k < j \leqslant p, \mathcal{J}, j \models \phi_1$

$(\Diamond^-(\mathsf{Start}(a) \wedge \Box^b \mathsf{executes}(a, b)))$, rather than whether it will continue to run this process forever. Moreover, we will see in Section 9.1 that using the bounded semantics can be of practical interest since it allows us to reduce TCQ answering in the presence of rigid predicates to TCQ answering without rigid predicates. We choose to keep both unbounded and bounded versions of future operators to cover the two settings that have been previously studied for TCQ answering.

The constraint that TCQs should not contain past operators nested under unbounded future operators will allow us to take advantage of the fact that a TKB entails the same BCQs for every time point $i > n$ to get a lower complexity in the case where there is no negation in the query. Indeed, in this case, the unknown future ($i > n$) can be entirely summarized in one time point $n + 1$. It will also be useful to get the data complexity upper bound of brave semantics in the case where there are no rigid predicates present nor negation in the queries. Moreover, it turns out that for some cases in our analysis, this restriction has no impact on our results. Indeed, [17] shows that Gabbay's separation theorem [41] can be used to rewrite a LTL formula ϕ containing bounded operators into a logically equivalent LTL formula ϕ' that is a Boolean combination of pure-past and pure-future formulas, although with an exponential blow-up. It follows that the restriction we impose does not have any influence over the data complexity of BTCQ entailment as long as negation is allowed in the query. Moreover, since past operators can still be nested under bounded future operators, all observations made can be referenced in the query language which can express most situations that could be desirable to detect.

It follows from the definition of certain answers that TCQ answering can be straightforwardly reduced to BTCQs entailment (using polynomially many tests w.r.t. data complexity and exponentially many tests w.r.t. combined complexity). For this reason, we focus on the latter problem.

Figure 2 summarizes the complexity of BTCQ entailment for DL-Lite$_\mathcal{R}$ and $\mathcal{EL}_\bot$ in the different cases depending on which kind of predicates are rigid. Our setting is slightly different from those of [6] and [22] because we have additional bounded operators and the restriction that past operators cannot be nested under unbounded future operators. However, the results shown in these papers apply to our setting. Indeed, the proofs for the lower bounds do not use past operators nested under future operators, and for the upper bounds, we argue that it is possible to reduce the entailment of a BTCQ ϕ that contains bounded operators to the entailment of a BTCQ ϕ' without bounded operators independently from the size of the TKB and linearly w.r.t. the query size. To do that, we add an assertion $\mathsf{end}(a)$ to the last ABox $\mathcal{A}_n$ of the sequence, where end and a are both fresh names, and rewrite the query without unbounded operators using the following equivalences: $\bullet^b\phi_1 \equiv \bigcirc\phi_1 \vee \mathsf{end}(a)$, $\Box^b\phi_1 \equiv \phi_1 \mathsf{U} (\mathsf{end}(a) \wedge \phi_1)$, $\Diamond^b\phi_1 \equiv \neg\mathsf{end}(a) \mathsf{U} \phi_1$, $\phi_1 \mathsf{U}^b \phi_2 \equiv (\phi_1 \wedge \neg\mathsf{end}(a)) \mathsf{U} \phi_2$.

3. Temporal query answering over inconsistent data

We extend the three inconsistency-tolerant semantics to temporal query answering. The main difference to the atemporal case is that in the presence of rigid concepts or roles, a TKB $\mathcal{K} = \langle \mathcal{T}, (\mathcal{A}_i)_{0 \leqslant i \leqslant n} \rangle$ may be inconsistent even if each KB $\langle \mathcal{T}, \mathcal{A}_i \rangle$ is consistent. Indeed, in this case there need not exist a sequence of interpretations $\mathcal{J} = (\mathcal{I}_i)_{i \geqslant 0}$ such that $\mathcal{I}_i$ is a model of $\langle \mathcal{T}, \mathcal{A}_i \rangle$ for every $i \in [\![0, n]\!]$ and which also respects rigid predicates. That is why we need to consider as repairs the $\mathcal{T}$-consistent sequences of subsets of the initial ABoxes that are component-wise maximal.

Definition 3.1 (Repair of a TKB)**.** A *repair* of a TKB $\mathcal{K} = \langle \mathcal{T}, (\mathcal{A}_i)_{0 \leqslant i \leqslant n} \rangle$ is a sequence of ABoxes $(\mathcal{A}'_i)_{0 \leqslant i \leqslant n}$ such that $\{(\alpha, i) \mid \alpha \in \mathcal{A}'_i, 0 \leqslant i \leqslant n\}$ is a maximal $\mathcal{T}$-consistent subset of $\{(\alpha, i) \mid \alpha \in \mathcal{A}_i,$

Rigid predicates	Data complexity		Combined complexity	
	DL-Lite$_{\mathcal{R}}$	$\mathcal{EL}_\bot$	DL-Lite$_{\mathcal{R}}$	$\mathcal{EL}_\bot$
$\mathsf{N_{RC}} = \mathsf{N_{RR}} = \emptyset$	ALOGTIME-complete	P-complete	PSPACE-complete	PSPACE-complete
$\mathsf{N_{RC}} \neq \emptyset, \mathsf{N_{RR}} = \emptyset$	ALOGTIME-complete	coNP-complete	PSPACE-complete	PSPACE-complete
$\mathsf{N_{RC}} \neq \emptyset, \mathsf{N_{RR}} \neq \emptyset$	ALOGTIME-complete	coNP-complete	PSPACE-complete	CONEXPTIME-complete

Figure 2. Complexity of BTCQ entailment in DL-Lite$_{\mathcal{R}}$ [6] and $\mathcal{EL}_\bot$ [22]

$0 \leqslant i \leqslant n\}$. We denote the set of repairs of $\mathcal{K}$ by $Rep(\mathcal{K})$.

The next example illustrates the impact rigid predicates can have on repairs.

Example 3.2. Consider the TKB $\mathcal{K} = \langle \mathcal{T}, (\mathcal{A}_i)_{0 \leqslant i \leqslant 1} \rangle$. The TBox expresses that web servers and application servers are two distinct kinds of servers, and the ABoxes provide information about a server a that executes two processes b and c.

$$\mathcal{T} = \{\mathsf{WebServer} \sqsubseteq \mathsf{Server},\ \mathsf{AppServer} \sqsubseteq \mathsf{Server},$$
$$\mathsf{WebServer} \sqsubseteq \neg\mathsf{AppServer}\}$$
$$\mathcal{A}_0 = \{\mathsf{WebServer}(a),\ \mathsf{executes}(a, b)\}$$
$$\mathcal{A}_1 = \{\mathsf{AppServer}(a), \mathsf{WebServer}(a), \mathsf{executes}(a, c)\}$$

Assume that no predicate is rigid. The TKB $\mathcal{K}$ is inconsistent because the timed assertions $(\mathsf{AppServer}(a), 1)$ and $(\mathsf{WebServer}(a), 1)$ violate the negative inclusion in $\mathcal{T}$. Specifically, $\mathsf{AppServer}(a)$ and $\mathsf{WebServer}(a)$ cannot both be true at time point 1. It follows that $\mathcal{K}$ has two repairs that correspond to the two different ways of restoring consistency: $(\mathcal{A}'_i)_{0 \leqslant i \leqslant 1}$ and $(\mathcal{A}''_i)_{0 \leqslant i \leqslant 1}$, where

$$\mathcal{A}'_0 = \mathcal{A}''_0 = \mathcal{A}_0$$
$$\mathcal{A}'_1 = \{\mathsf{AppServer}(a), \mathsf{executes}(a, c)\}$$
$$\mathcal{A}''_1 = \{\mathsf{WebServer}(a), \mathsf{executes}(a, c)\}.$$

Now assume that $\mathsf{AppServer}$ is rigid. There is then a new reason for $\mathcal{K}$ being inconsistent: the timed assertions $(\mathsf{WebServer}(a), 0)$ and $(\mathsf{AppServer}(a), 1)$ violate the negative inclusion of $\mathcal{T}$ due to the rigidity of $\mathsf{AppServer}$, which implies that $\mathsf{AppServer}(a)$ and $\mathsf{WebServer}(a)$ should be both entailed at time point 0. Therefore, $\mathcal{K}$ has now the two repairs $(\mathcal{A}'_i)_{0 \leqslant i \leqslant 1}$ and $(\mathcal{A}''_i)_{0 \leqslant i \leqslant 1}$, where

$$\mathcal{A}'_0 = \{\mathsf{executes}(a, b)\}$$
$$\mathcal{A}'_1 = \{\mathsf{AppServer}(a), \mathsf{executes}(a, c)\}$$
$$\mathcal{A}''_0 = \mathcal{A}_0$$
$$\mathcal{A}''_1 = \{\mathsf{WebServer}(a), \mathsf{executes}(a, c)\}.$$

Note that even though $(\mathcal{A}'_i)_{0 \leqslant i \leqslant 1}$ is maximal (adding $\mathsf{WebServer}(a)$ to $\mathcal{A}'_0$ renders the TKB inconsistent), $\mathcal{A}'_0$ is not a repair of $\langle \mathcal{T}, \mathcal{A}_0 \rangle$, because it is not maximal.

We extend the semantics AR, IAR, and brave to the temporal case in the natural way by regarding sequences of ABoxes.

Definition 3.3 (AR, IAR, brave semantics for TCQs)**.** A tuple $\vec{a}$ is an *answer* to a TCQ ϕ over a TKB $\mathcal{K} = \langle \mathcal{T}, (\mathcal{A}_i)_{0 \leqslant i \leqslant n} \rangle$ at time point p under

- AR semantics, written $\mathcal{K}, p \models_{\text{AR}} \phi(\vec{a})$, iff $\langle \mathcal{T}, (\mathcal{A}'_i)_{0 \leqslant i \leqslant n} \rangle, p \models \phi(\vec{a})$ for every repair $(\mathcal{A}'_i)_{0 \leqslant i \leqslant n}$ of $\mathcal{K}$;
- IAR semantics, written $\mathcal{K}, p \models_{\text{IAR}} \phi(\vec{a})$, iff $\langle \mathcal{T}, (\mathcal{A}^{\text{ir}}_i)_{0 \leqslant i \leqslant n} \rangle, p \models \phi(\vec{a})$, with $\mathcal{A}^{\text{ir}}_i = \bigcap_{(\mathcal{A}'_j)_{0 \leqslant j \leqslant n} \in Rep(\mathcal{K})} \mathcal{A}'_i$ for all $i \in [\![0, n]\!]$;
- brave semantics, written $\mathcal{K}, p \models_{\text{brave}} \phi(\vec{a})$, iff $\langle \mathcal{T}, (\mathcal{A}'_i)_{0 \leqslant i \leqslant n} \rangle, p \models \phi(\vec{a})$ for some repair $(\mathcal{A}'_i)_{0 \leqslant i \leqslant n}$ of $\mathcal{K}$.

The following relationships between the semantics, which already hold in the atemporal case, are implied by their definition:

$$\mathcal{K}, p \models_{\text{IAR}} \phi(\vec{a}) \Rightarrow \mathcal{K}, p \models_{\text{AR}} \phi(\vec{a}) \Rightarrow \mathcal{K}, p \models_{\text{brave}} \phi(\vec{a})$$

We illustrate the effect of the different semantics in the temporal case in the following example.

Example 3.4 (Example 3.2 cont'd)**.** Consider the following three temporal conjunctive queries.

$$\phi_1 = \Box^b(\exists y.\mathsf{executes}(x, y))$$
$$\phi_2 = \Box^b(\exists y.\mathsf{Server}(x) \land \mathsf{executes}(x, y))$$
$$\phi_3 = \Box^b(\exists y.\mathsf{AppServer}(x) \land \mathsf{executes}(x, y))$$

If there are no rigid predicates, the intersection of the repairs is $(\mathcal{A}^{\text{ir}}_i)_{0 \leqslant i \leqslant 1}$, with $\mathcal{A}^{\text{ir}}_0 = \mathcal{A}_0$ and $\mathcal{A}^{\text{ir}}_1 =$

$\{\mathsf{executes}(a,c)\}$. We have $\mathcal{K},0 \models_{\text{IAR}} \phi_1(a)$, because in every model of the intersection of the repairs a executes b at time point 0 and c at time point 1. For ϕ_2, $\mathcal{K},0 \models_{\text{AR}} \phi_2(a)$, since every model of every repair assigns a to $\mathsf{WebServer}$ at time point 0 and to either $\mathsf{AppServer}$ (in models of $(\mathcal{A}'_i)_{0\leqslant i\leqslant 1}$) or $\mathsf{WebServer}$ (in models of $(\mathcal{A}''_i)_{0\leqslant i\leqslant 1}$) at time point 1. However, $\mathcal{K},0 \not\models_{\text{IAR}} \phi_2(a)$. Finally, $\mathcal{K},0 \not\models_{\text{brave}} \phi_3(a)$, because no repair entails $\mathsf{AppServer}(a)$ at time point 0.

If $\mathsf{AppServer}$ is rigid, the intersection of the repairs is $(\mathcal{A}^{\text{ir}}_i)_{0\leqslant i\leqslant 1}$ with $\mathcal{A}^{\text{ir}}_0 = \{\mathsf{executes}(a,b)\}$ and $\mathcal{A}^{\text{ir}}_1 = \{\mathsf{executes}(a,c)\}$. So, still $\mathcal{K},0 \models_{\text{IAR}} \phi_1(a)$. Since every model of every repair assigns a to Server at time points 0 and 1 (either because a is a web server or because a is an application server), $\mathcal{K},0 \models_{\text{AR}} \phi_2(a)$, but $\mathcal{K},0 \not\models_{\text{IAR}} \phi_2(a)$. Finally, $\mathcal{K},0 \models_{\text{brave}} \phi_3(a)$, because every model of $\langle\mathcal{T},(\mathcal{A}'_i)_{0\leqslant i\leqslant 1}\rangle$ assigns a to $\mathsf{AppServer}$ at any time point by rigidity of $\mathsf{AppServer}$, but $\mathcal{K},0 \not\models_{\text{AR}} \phi_3(a)$.

We conclude this section by pointing out some characteristics of the case without rigid predicates that will be useful later. If there are no rigid predicates, the interpretations $\mathcal{I}_i$ of a model $\mathcal{J} = (\mathcal{I}_i)_{i\geqslant 0}$ of $\mathcal{K}$ that respects rigid predicates are independent, besides the interpretation of the individual names. We thus obtain the following proposition.

Proposition 3.5. *If $\mathsf{N_{RC}} = \mathsf{N_{RR}} = \emptyset$, then a TKB $\mathcal{K} = \langle\mathcal{T},(\mathcal{A}_i)_{0\leqslant i\leqslant n}\rangle$ is consistent iff every $\langle\mathcal{T},\mathcal{A}_i\rangle$ is consistent. Moreover, if $\mathcal{K}$ is consistent, for every $p \in [\![0,n]\!]$, $\mathcal{I}'_p$ is a model of $\langle\mathcal{T},\mathcal{A}_p\rangle$ iff there exists a model $\mathcal{J} = (\mathcal{I}_i)_{i\geqslant 0}$ of $\mathcal{K}$ such that $\mathcal{I}_p = \mathcal{I}'_p$.*

Proof. If $\mathsf{N_{RC}} = \mathsf{N_{RR}} = \emptyset$, a sequence of interpretations $\mathcal{J} = (\mathcal{I}_i)_{i\geqslant 0}$ is a model of $\mathcal{K}$ that respects rigid predicates iff it is a model of $\mathcal{K}$. It follows that $\mathcal{K}$ is consistent iff there exists $\mathcal{J} = (\mathcal{I}_i)_{i\geqslant 0}$ such that for every $i \in [\![0,n]\!]$, $\mathcal{I}_i$ is a model of $\langle\mathcal{T},\mathcal{A}_i\rangle$, for every $i > n$, $\mathcal{I}_i$ is a model of $\mathcal{T}$, and for every $a \in \mathsf{N_I}$ and all $i,j \geqslant 0$, $a^{\mathcal{I}_i} = a^{\mathcal{I}_j}$. We show that this is the case iff each $\langle\mathcal{T},\mathcal{A}_i\rangle$ has a model. Let $\mathcal{I}'_0 = \langle\Delta^{\mathcal{I}'_0},\cdot^{\mathcal{I}'_0}\rangle,\dots,\mathcal{I}'_n = \langle\Delta^{\mathcal{I}'_n},\cdot^{\mathcal{I}'_n}\rangle$ be models of $\langle\mathcal{T},\mathcal{A}_0\rangle,\dots\langle\mathcal{T},\mathcal{A}_n\rangle$ respectively, and $p \in [\![0,n]\!]$. For every $i \in [\![0,n]\!]$, let $\mathcal{I}_i = \langle\Delta,\cdot^{\mathcal{I}_i}\rangle$, where $\Delta = \Delta^{\mathcal{I}'_p}$, and $\cdot^{\mathcal{I}_i}$ is defined as follows: $a^{\mathcal{I}_i} = a^{\mathcal{I}'_p}$ for every $a \in \mathsf{N_I}$, $A^{\mathcal{I}_i} = \{a^{\mathcal{I}'_p} \mid a^{\mathcal{I}'_i} \in A^{\mathcal{I}'_i}\}$ for every $A \in \mathsf{N_C}$, and $R^{\mathcal{I}_i} = \{(a^{\mathcal{I}'_p},b^{\mathcal{I}'_p}) \mid (a^{\mathcal{I}'_i},b^{\mathcal{I}'_i}) \in R^{\mathcal{I}'_i}\}$ for every $R \in \mathsf{N_R}$. Since we adopted the unique name assumption, each $\mathcal{I}_i$ is a model of $\langle\mathcal{T},\mathcal{A}_i\rangle$. It follows that $\mathcal{J} = (\mathcal{I}_i)_{i\geqslant 0}$ with $\mathcal{I}_i = \emptyset$ for $i > n$ is such that for every $i \in [\![0,n]\!]$, $\mathcal{I}_i$ is a model of $\langle\mathcal{T},\mathcal{A}_i\rangle$, for every $i > n$, $\mathcal{I}_i$ is a model of $\mathcal{T}$, and for every $a \in \mathsf{N_I}$ and all $i,j \geqslant 0$, $a^{\mathcal{I}_i} = a^{\mathcal{I}_j}$. Moreover, $\mathcal{J}$ is such that $\mathcal{I}_p = \mathcal{I}'_p$. The other direction is trivial since $\mathcal{I}_p$ is a model of $\langle\mathcal{T},\mathcal{A}_p\rangle$. □

It follows that CQs can be answered at time point p by answering them over the KB $\langle\mathcal{T},\mathcal{A}_p\rangle$.

Proposition 3.6. *If $\mathsf{N_{RC}} = \mathsf{N_{RR}} = \emptyset$, then for every BCQ $q = \exists\vec{y}.\psi(\vec{y})$ and $p \in [\![0,n]\!]$, $\mathcal{K},p \models q$ iff $\langle\mathcal{T},\mathcal{A}_p\rangle \models q$.*

Proof. $\mathcal{K},p \models q$ iff for every model $\mathcal{J} = (\mathcal{I}_i)_{i\geqslant 0}$ of $\mathcal{K}$ that respects rigid predicates, $\mathcal{I}_p \models q$. By Proposition 3.5, this is the case iff for every model $\mathcal{I}_p$ of $\langle\mathcal{T},\mathcal{A}_p\rangle$, $\mathcal{I}_p \models q$, which is equivalent to $\langle\mathcal{T},\mathcal{A}_p\rangle \models q$. □

4. Complexity analysis overview

In the next four sections, we investigate the complexity of inconsistency-tolerant BTCQ entailment in DL-Lite$_\mathcal{R}$ and $\mathcal{EL}_\bot$. Apart from the different DLs, we also consider two settings of query languages: in the first setting, all TCQs as defined in Section 2 are considered, in the second setting, we analyze the complexity for TCQs that do not use any negation operators. For classical semantics, some complexities have been investigated earlier for the different settings we consider. For the case where *negations are allowed* in the queries, the complexity of BTCQ entailment under the classical semantics has been studied in [6] for DL-Lite$_\mathcal{R}$ and in [22] for $\mathcal{EL}$ (cf. Section 2, Figure 2). Furthermore, it has also been shown in [17, 42] that in DL-Lite$_\mathcal{R}$, TCQs *without negation* (and with a bounded future semantics) can be rewritten into FO-queries for temporal databases, but only for a restricted framework without rigid roles and with rigid concepts only for TCQs that are rooted. We follow a similar route in this paper and consider TCQs with and without negations explicitly, for which we analyze the data and combined complexity for $\mathcal{EL}_\bot$ and DL-Lite$_\mathcal{R}$ TKBs.

Most of our complexity upper bounds are based on a set of general algorithms for BTCQ entailment under the different inconsistency-tolerant semantics, which we present in Section 5. Those allow to deduce complexity upper bounds for the different settings based on the complexity of BTCQ entailment under classical semantics, on the complexity of recognizing repairs, and on the complexity of consistency checking. In Section 6, we establish the complexity of these basic tasks, and give complexity results for our two DLs of interest, $\mathcal{EL}_\bot$ and DL-Lite$_\mathcal{R}$, regarding both data and combined complexity. In the cases where the general algorithms are insufficient to give tight bounds, we try to provide specialized algorithms.

We then study the complexity of the entailment of BTCQs without negation. In Section 7, we first investigate this case under classical semantics, and observe that in some cases, disallowing negations leads to lower worst case complexities, even if we alleviate the limitations imposed in [17, 42]. These lower complexities allow us to also improve the complexity bounds for inconsistency-tolerant reasoning when there are no negations in the TCQs in Section 8. Furthermore, we take advantage of the absence of negations to tighten an upper bound for brave semantics without rigid predicates.

We recall the definitions of the complexity classes that appear in this section.

- P: problems solvable in polynomial time.
- NP: problems solvable in non-deterministic polynomial time.
- coNP: problems whose complement is in NP.
- $\Delta_2^p[O(\log n)]$: problems solvable in polynomial time with at most logarithmically many calls to an NP oracle.
- Σ_2^p: problems solvable in non-deterministic polynomial time with an NP oracle.
- Π_2^p: problems whose complement is in Σ_2^p.
- AC^0: problems that can be solved by a uniform family of circuits of constant depth and polynomial size, with unbounded-fanin AND and OR gates. We have $\mathsf{AC}^0 \subseteq$ P.
- ALOGTIME: problems solvable in logarithmic time by a random access alternating Turing machine. We have $\mathsf{AC}^0 \subseteq$ ALOGTIME $\subseteq$ P.
- PSPACE: problems solvable in polynomial space.
- EXPTIME: problems solvable in exponential time.
- NEXPTIME: problems which are solvable in non-deterministic exponential time.
- CONEXPTIME: problems whose complement is in NEXPTIME.

For the remainder of this paper, $\mathcal{L}$ is a DL which is interpreted w.r.t. standard interpretations, as defined in Section 2. We will consider in particular the cases $\mathcal{L}$ = DL-Lite$_\mathcal{R}$ and $\mathcal{L} = \mathcal{EL}_\bot$. Furthermore, we assume $\mathcal{K} = \langle \mathcal{T}, (\mathcal{A}_i)_{0\leqslant i\leqslant n}\rangle$ to be the TKB we evaluate our query against, and ϕ to be the considered query.

5. General algorithms for inconsistency-tolerant BTCQ entailment

Our complexity bounds are based on a set of general algorithms for deciding BTCQ entailment under the different semantics, which are inspired from known algorithms for inconsistency-tolerant BCQ entailment in the atemporal case (see e.g., [34]).

Non-Entailment under AR semantics. The procedure $\mathsf{ARNonEntailment}$ decides whether ϕ is *not* entailed by $\mathcal{K}$ at time point p under AR semantics, and is defined as follows.

1. Guess a sequence of ABoxes $(\mathcal{A}'_i)_{0\leqslant i\leqslant n} \subseteq (\mathcal{A}_i)_{0\leqslant i\leqslant n}$.
2. Verify that $(\mathcal{A}'_i)_{0\leqslant i\leqslant n}$ is a repair of $\mathcal{K}$ and that $\langle \mathcal{T}, (\mathcal{A}'_i)_{0\leqslant i\leqslant n}\rangle, p \not\models \phi$.

Entailment under brave semantics. The procedure $\mathsf{braveEntailment}$ decides whether ϕ is entailed by $\mathcal{K}$ at time point p under brave semantics, and is defined as follows.

1. Guess a sequence of ABoxes $(\mathcal{A}'_i)_{0\leqslant i\leqslant n} \subseteq (\mathcal{A}_i)_{0\leqslant i\leqslant n}$.
2. Verify that $(\mathcal{A}'_i)_{0\leqslant i\leqslant n}$ is a repair of $\mathcal{K}$ and that $\langle \mathcal{T}, (\mathcal{A}'_i)_{0\leqslant i\leqslant n}\rangle, p \models \phi$.

Non-Entailment under IAR semantics. The procedure $\mathsf{IARNonEntailment}$ decides whether ϕ is *not* entailed by $\mathcal{K}$ at time point p under IAR semantics, and is defined as follows.

1. Guess
   (a) a set $\mathcal{B} = \{(\alpha_1, i_1), \dots, (\alpha_m, i_m)\} \subseteq (\mathcal{A}_i)_{0\leqslant i\leqslant n}$ of timed assertions, together with
   (b) m subsets of the data $(\mathcal{A}'^1_i)_{0\leqslant i\leqslant n} \subseteq (\mathcal{A}_i)_{0\leqslant i\leqslant n}$, $\dots$, $(\mathcal{A}'^m_i)_{0\leqslant i\leqslant n} \subseteq (\mathcal{A}_i)_{0\leqslant i\leqslant n}$ such that for every $j \in [\![1, m]\!]$, $\alpha_j \notin \mathcal{A}'^j_{i_j}$.
2. Verify that
   (a) for every $j \in [\![1, m]\!]$, $(\mathcal{A}'^j_i)_{0\leqslant i\leqslant n}$ is a repair of $\mathcal{K}$, and
   (b) $\langle \mathcal{T}, (\mathcal{A}_i)_{0\leqslant i\leqslant n} \setminus \mathcal{B}\rangle, p \not\models \phi$.

Note that, since $m \leqslant |(\mathcal{A}_i)_{0\leqslant i\leqslant n}|$, Step 2a has to verify only a linear number of repairs. We show that the algorithm decides non-entailment under IAR semantics. Indeed, if for every (α_j, i_j) there exists a repair $(\mathcal{A}'^j_i)_{0\leqslant i\leqslant n}$ of $\mathcal{K}$ that does not contain (α_j, i_j), then (α_j, i_j) is not in the intersection of the repairs of $\mathcal{K}$. Thus $(\mathcal{A}_i)_{0\leqslant i\leqslant n} \setminus \{(\alpha_1, i_1), \dots, (\alpha_m, i_m)\}$ is a superset of the intersection $(\mathcal{A}^{\mathrm{ir}}_i)_{0\leqslant i\leqslant n}$ of the repairs of $\mathcal{K}$. It follows that if $\langle \mathcal{T}, (\mathcal{A}_i)_{0\leqslant i\leqslant n} \setminus \mathcal{B}\rangle, p \not\models \phi$, then $\mathcal{K}, p \not\models_{\mathrm{IAR}} \phi$. In the other direction, assume that

$\mathcal{K}, p \not\models_{\text{IAR}} \phi$, and let $\mathcal{B} = \{(\alpha_1, i_1), \dots, (\alpha_m, i_m)\} = (\mathcal{A}_i)_{0\leqslant i\leqslant n} \setminus (\mathcal{A}_i^{\text{ir}})_{0\leqslant i\leqslant n}$. For each (α_j, i_j), there exists a repair $(\mathcal{A}_i'^{j})_{0\leqslant i\leqslant n}$ of $\mathcal{K}$ that does not contain the timed assertion (α_j, i_j), and $\langle \mathcal{T}, (\mathcal{A}_i)_{0\leqslant i\leqslant n} \setminus \mathcal{B}\rangle, p \not\models \phi$.

Entailment under IAR semantics. We give an alternative procedure for IAR, IAREntailment, which decides whether ϕ is entailed under IAR semantics at time point p.

1. Compute the maximal size k_{max} of a minimal $\mathcal{T}$-inconsistent subset of $(\mathcal{A}_i)_{0\leqslant i\leqslant n}$ by binary search, asking an oracle if there exists a $\mathcal{T}$-inconsistent set of timed assertions $\mathcal{B} \subseteq (\mathcal{A}_i)_{0\leqslant i\leqslant n}$ such that for every $(\alpha, i) \in \mathcal{B}$, $\mathcal{B} \setminus \{(\alpha, i)\}$ is consistent and $|\mathcal{B}| \geqslant k$, where k is the input. Note that verifying whether a candidate fulfils these conditions requires only a polynomial number of consistency checks.
2. Compute the minimal inconsistent subsets of $\mathcal{K}$ by checking the consistency of every subset of $(\mathcal{A}_i)_{0\leqslant i\leqslant n}$ of size at most k_{max}.
3. Call the oracle to determine whether ϕ is entailed at time point p by the TKB from which the minimal inconsistent subsets have been removed.

We show that the intersection of the repairs of $\mathcal{K}$ is obtained by removing the minimal inconsistent subsets of $\mathcal{K}$. Let $\mathcal{B} \subseteq (\mathcal{A}_i)_{0\leqslant i\leqslant n}$ be a minimal inconsistent subset of $\mathcal{K}$ and $(\alpha, i) \in \mathcal{B}$. Since $\mathcal{B} \setminus \{(\alpha, i)\}$ is consistent, (α, i) is not in the repairs that contain $\mathcal{B} \setminus \{(\alpha, i)\}$. In the other direction, if a timed assertion (α, i) does not appear in some repair $(\mathcal{A}_i')_{0\leqslant i\leqslant n}$ of $\mathcal{K}$, since the repairs are maximal, $(\mathcal{A}_i')_{0\leqslant i\leqslant n} \cup \{(\alpha, i)\}$ is inconsistent so (α, i) is in some minimal inconsistent subset of $\mathcal{K}$.

6. Complexity of inconsistency-tolerant BTCQ entailment with negations in the query

In this section, we investigate the complexity of BTCQ entailment for general BTCQs, that is, BTCQs that may contain negations. For this, we first establish the complexities of consistency checking and repair recognition, i.e., the task of deciding whether a sequence of ABoxes is a repair of $\mathcal{K}$. We then build on these results to prove the complexity of inconsistency-tolerant temporal query entailment using the algorithms described in the last section, while showing matching lower bounds. In order to show data complexity bounds for DL-Lite$_\mathcal{R}$ that are below P, we follow a different route, by defining an abstract structure that captures BTCQ entailment under IAR and brave semantics and can be verified in ALogTime. We thus obtain the following theorem.

Theorem 6.1. *The results in Figure 3 hold.*

6.1. Consistency checking and repair recognition for TKBs

We reduce these tasks to the atemporal case by defining an atemporal KB $\widetilde{\mathcal{K}}$ based on $\mathcal{K}$. For $\mathcal{K} = \langle \mathcal{T}, (\mathcal{A}_i)_{0\leqslant i\leqslant n}\rangle$, $\widetilde{\mathcal{K}}$ is defined in Figure 4. We first show a correspondence between the models of $\mathcal{K}$ that respect rigid predicates and the models of $\widetilde{\mathcal{K}}$.

Lemma 6.2. *$\mathcal{K}$ is consistent iff $\widetilde{\mathcal{K}}$ is consistent.*

Proof. ($\Leftarrow$) We construct a function temp from the models of $\widetilde{\mathcal{K}}$ to those of $\mathcal{K}$ that respect rigid predicates. Assume $\widetilde{\mathcal{K}}$ is consistent, and let $\widetilde{\mathcal{I}}$ be a model of $\langle \widetilde{\mathcal{T}}, \widetilde{\mathcal{A}}\rangle$. We define $\mathsf{temp}(\widetilde{\mathcal{I}}) = \mathcal{J} = (\mathcal{I}_i)_{i\geqslant 0}$ as follows. For every $i \in [\![0, n]\!]$, we set

- $a^{\mathcal{I}_i} = a^{\widetilde{\mathcal{I}}}$ for every $a \in \mathsf{N_I}$,
- $A^{\mathcal{I}_i} = A^{\widetilde{\mathcal{I}}}$ for every $A \in \mathsf{N_{RC}}$,
- $R^{\mathcal{I}_i} = R^{\widetilde{\mathcal{I}}}$ for every $R \in \mathsf{N_{RR}}$,
- $A^{\mathcal{I}_i} = A_i^{\widetilde{\mathcal{I}}}$ for every $A \in \mathsf{N_C} \setminus \mathsf{N_{RC}}$, and
- $R^{\mathcal{I}_i} = R_i^{\widetilde{\mathcal{I}}}$ for every $R \in \mathsf{N_R} \setminus \mathsf{N_{RR}}$,

and for every $i > n$, we set

- $a^{\mathcal{I}_i} = a^{\widetilde{\mathcal{I}}}$ for every $a \in \mathsf{N_I}$,
- $A^{\mathcal{I}_i} = A^{\widetilde{\mathcal{I}}}$ for every $A \in \mathsf{N_{RC}}$,
- $R^{\mathcal{I}_i} = R^{\widetilde{\mathcal{I}}}$ for every $R \in \mathsf{N_{RR}}$,
- $A^{\mathcal{I}_i} = A_{n+1}^{\widetilde{\mathcal{I}}}$ for every $A \in \mathsf{N_C} \setminus \mathsf{N_{RC}}$, and
- $R^{\mathcal{I}_i} = R_{n+1}^{\widetilde{\mathcal{I}}}$ for every $R \in \mathsf{N_R} \setminus \mathsf{N_{RR}}$.

We show that the sequence of interpretations $\mathsf{temp}(\widetilde{\mathcal{I}})$ is a model of $\mathcal{K}$ that respects rigid predicates.

1. For every $i \in [\![0, n]\!]$, $\mathcal{I}_i$ is a model of $\mathcal{A}_i$. If $A(a) \in \mathcal{A}_i$, then either $A \in \mathsf{N_{RC}}$ and $A(a) \in \widetilde{\mathcal{A}}$, or $A \notin \mathsf{N_{RC}}$ and $A_i(a) \in \widetilde{\mathcal{A}}$. In both cases, $a^{\mathcal{I}_i} = a^{\widetilde{\mathcal{I}}} \in A^{\mathcal{I}_i}$. We can argue in the same way for the role assertions in $\mathcal{A}_i$.
2. For every $i \in [\![0, n+1]\!]$, $\mathcal{I}_i$ is a model of $\mathcal{T}$. Slightly abusing notation, we denote by $\mathsf{RenameNotRig}(\mathcal{I}_i, i)$ the interpretation obtained from $\mathcal{I}_i$ by renaming every non-rigid predicate X by X_i. (We can see an interpretation as an infinite set of assertions.) The interpretations of all rigid predicates and of all A_i and R_i are the same in $\mathsf{RenameNotRig}(\mathcal{I}_i, i)$ and $\widetilde{\mathcal{I}}$. Since $\widetilde{\mathcal{I}}$ is a model of $\widetilde{\mathcal{T}}$, and $\widetilde{\mathcal{T}}$ does not con-

	Data complexity				Combined complexity			
	classical	AR	IAR	brave	classical	AR	IAR	brave
$\mathcal{EL}_\bot$								
$\mathsf{N_{RC}} = \mathsf{N_{RR}} = \emptyset$	P	coNP	coNP	NP	PSPACE	PSPACE	PSPACE	PSPACE
$\mathsf{N_{RC}} \neq \emptyset, \mathsf{N_{RR}} = \emptyset$	coNP	coNP	coNP	Σ_2^p	PSPACE	PSPACE	PSPACE	PSPACE
$\mathsf{N_{RC}} \neq \emptyset, \mathsf{N_{RR}} \neq \emptyset$	coNP	coNP	coNP	Σ_2^p	coNEXPTIME	coNEXPTIME	coNEXPTIME	coNEXPTIME
DL-Lite$_\mathcal{R}$								
$\mathsf{N_{RC}} = \mathsf{N_{RR}} = \emptyset$	ALOGTIME	coNP	in P	in NP	PSPACE	PSPACE	PSPACE	PSPACE
$\mathsf{N_{RC}} \neq \emptyset, \mathsf{N_{RR}} = \emptyset$	ALOGTIME	coNP	in P	NP	PSPACE	PSPACE	PSPACE	PSPACE
$\mathsf{N_{RC}} \neq \emptyset, \mathsf{N_{RR}} \neq \emptyset$	ALOGTIME	coNP	in P	NP	PSPACE	PSPACE	PSPACE	PSPACE

Figure 3. Data [left] and combined [right] complexity of BTCQ entailment for BTCQs with negations. All complexities are tight, except those preceded by "in", which are upper bounds. The results for the classical semantics come from [22] for $\mathcal{EL}_\bot$ and [6] for DL-Lite$_\mathcal{R}$.

$$\widetilde{\mathcal{T}} = \bigcup_{i=0}^{n+1} \mathsf{RenameNotRig}(\mathcal{T}, i)$$

$$\widetilde{\mathcal{A}} = \bigcup_{i=0}^{n} \mathsf{RenameNotRig}(\mathcal{A}_i, i)$$

where for every set of axioms $\mathcal{O}$, the function $\mathsf{RenameNotRig}(\mathcal{O}, i)$ substitutes every non-rigid predicate X by X_i in every axiom $\alpha \in \mathcal{O}$.

Figure 4. KB $\widetilde{\mathcal{K}} = \langle \widetilde{\mathcal{T}}, \widetilde{\mathcal{A}} \rangle$ representing $\mathcal{K} = \langle \mathcal{T}, (\mathcal{A}_i)_{0 \leqslant i \leqslant n} \rangle$.

tain any axiom that involves two non-rigid predicates X_i and X_j with $i \neq j$, $\mathsf{RenameNotRig}(\mathcal{I}_i, i)$ is a model of $\widetilde{\mathcal{T}}$. Moreover $\mathsf{RenameNotRig}(\mathcal{T}, i) \subseteq \widetilde{\mathcal{T}}$, and therefore $\mathsf{RenameNotRig}(\mathcal{I}_i, i)$ is a model of $\mathsf{RenameNotRig}(\mathcal{T}, i)$. Hence, $\mathcal{I}_i$ is a model of $\mathcal{T}$.

3. For every $i > n+1$, $\mathcal{I}_i = \mathcal{I}_{n+1}$ is a model of $\mathcal{T}$.
4. For every $i \geqslant 0$, for every $A \in \mathsf{N_{RC}}$, $A^{\mathcal{I}_i} = A^{\widetilde{\mathcal{I}}}$, and for every $R \in \mathsf{N_{RC}}$, $R^{\mathcal{I}_i} = R^{\widetilde{\mathcal{I}}}$. Therefore, $\mathcal{J}$ respects rigid predicates.

We obtain that $\mathsf{temp}(\widetilde{\mathcal{I}})$ is a model of $\mathcal{K}$ that respects rigid predicates.

($\Rightarrow$) For the other direction, we construct a function atemp from the models of $\mathcal{K}$ that respect rigid predicates to those of $\widetilde{\mathcal{K}}$. Assume $\mathcal{K}$ is consistent, and let $\mathcal{J} = (\mathcal{I}_i)_{i \geqslant 0}$ be a model of $\mathcal{K}$ that respects rigid predicates. We define $\mathsf{atemp}(\mathcal{J}) = \widetilde{\mathcal{I}}$ as follows.

– $a^{\widetilde{\mathcal{I}}} = a^{\mathcal{I}_0}$ $(= a^{\mathcal{I}_i}$ for every $i \geqslant 0)$ for every $a \in \mathsf{N_I}$,
– $A^{\widetilde{\mathcal{I}}} = A^{\mathcal{I}_0}$ for every $A \in \mathsf{N_{RC}}$,
– $R^{\widetilde{\mathcal{I}}} = R^{\mathcal{I}_0}$ for every $R \in \mathsf{N_{RR}}$,
– $A_i^{\widetilde{\mathcal{I}}} = A^{\mathcal{I}_i}$ for every $A \in \mathsf{N_C} \backslash \mathsf{N_{RC}}$ and $i \in [\![0, n]\!]$, and
– $R_i^{\widetilde{\mathcal{I}}} = R^{\mathcal{I}_i}$ for every $R \in \mathsf{N_R} \backslash \mathsf{N_{RR}}$ and $i \in [\![0, n]\!]$.

Again, we show that $\widetilde{\mathcal{I}}$ is a model of $\widetilde{\mathcal{K}}$ by considering the ABox and the TBox separately.

1. $\widetilde{\mathcal{I}}$ is a model of $\widetilde{\mathcal{A}}$. If $A(a) \in \widetilde{\mathcal{A}}$ with $A \in \mathsf{N_{RC}}$, then $a^{\widetilde{\mathcal{I}}} = a^{\mathcal{I}_0} \in A^{\widetilde{\mathcal{I}}}$, and if $A_i(a) \in \widetilde{\mathcal{A}}$ for some $A \notin \mathsf{N_{RC}}$, then $A(a) \in \mathcal{A}_i$ and $a^{\widetilde{\mathcal{I}}} = a^{\mathcal{I}_i} \in A^{\widetilde{\mathcal{I}}}$. The situation is the same for the role assertions in $\widetilde{\mathcal{A}}$.
2. $\widetilde{\mathcal{I}}$ is a model of $\widetilde{\mathcal{T}}$. If we rename the non-rigid predicates, $\mathsf{RenameNotRig}(\mathcal{I}_i, i)$ coincides with $\widetilde{\mathcal{I}}$ on the interpretation of all rigid predicates and all A_i and R_i. Since each $\mathcal{I}_i$ is a model of $\mathcal{T}$, each interpretation $\mathsf{RenameNotRig}(\mathcal{I}_i, i)$ is a model of $\mathsf{RenameNotRig}(\mathcal{T}, i)$, and since $\widetilde{\mathcal{T}}$ does not contain any axiom that involve two non-rigid predicates X_i and X_j with $i \neq j$, each $\mathsf{RenameNotRig}(\mathcal{I}_i, i)$ is a model of $\widetilde{\mathcal{T}}$. It follows that $\widetilde{\mathcal{I}}$ is a model of $\widetilde{\mathcal{T}}$.

We thus shown a direct correspondence between the models of $\mathcal{K}$ and those of $\widetilde{\mathcal{K}}$, and obtain that $\mathcal{K}$ is satisfiable iff $\widetilde{\mathcal{K}}$ is satisfiable. □

It follows that consistency checking of TKBs can be polynomially reduced to consistency checking of KBs.

Lemma 6.3. *If for a DL $\mathcal{L}$, consistency checking of $\mathcal{L}$ KBs is in* P*, then consistency checking of $\mathcal{L}$ TKBs is in* P *as well.*

Proof. By Lemma 6.2, the TKB $\mathcal{K}$ is consistent iff the KB $\widetilde{\mathcal{K}}$ is consistent. If consistency checking is in P for $\mathcal{L}$ KBs, the consistency of $\mathcal{K}$ can then be checked in time polynomial in the size of $\widetilde{\mathcal{T}}$ and $\widetilde{\mathcal{A}}$. Since the size of $\widetilde{\mathcal{T}}$ is polynomial in $|\mathcal{T}|$ and n, and the size of $\widetilde{\mathcal{A}}$ is at most the size of $(\mathcal{A}_i)_{0\leqslant i\leqslant n}$, we obtain that TKB consistency checking is in P. □

We next show that repair recognition can be done with a polynomial number of consistency checks.

Lemma 6.4. *If for a DL $\mathcal{L}$, consistency checking of $\mathcal{L}$ TKBs is in* P*, then repair recognition, i.e., deciding whether a sequence of ABoxes $(\mathcal{A}'_i)_{0\leqslant i\leqslant n}$ is a repair of $\mathcal{K}$, is in* P.

Proof. Assume consistency checking of $\mathcal{L}$ TKBs is in P. Then, we can verify in P whether a sequence of ABoxes $(\mathcal{A}'_i)_{0\leqslant i\leqslant n}$ is a repair of $\mathcal{K}$ as follows.

1. For every i, check that $\mathcal{A}'_i \subseteq \mathcal{A}_i$.
2. Check that $(\mathcal{A}'_i)_{0\leqslant i\leqslant n}$ is $\mathcal{T}$-consistent.
3. For every $(\alpha, j) \in (\mathcal{A}_i)_{0\leqslant i\leqslant n}\setminus(\mathcal{A}'_i)_{0\leqslant i\leqslant n}$, check that $(\mathcal{A}'_i)_{0\leqslant i\leqslant n} \cup \{(\alpha, j)\}$ is $\mathcal{T}$-inconsistent. □

Note that Lemmas 6.3 and 6.4 apply to DL-Lite$_\mathcal{R}$ and $\mathcal{EL}_\bot$.

6.2. Combined complexity

We now are ready to establish the complexity of BTCQ entailment under inconsistency-tolerant semantics. We start with the combined complexity. The following upper bounds follow straightforwardly from the procedures described in Section 5.

Proposition 6.5. *If repair recognition is in* P *and BTCQ entailment under the classical semantics is in* PSPACE *w.r.t. combined complexity, then BTCQ entailment under AR, IAR and brave semantics is in* PSPACE *w.r.t. combined complexity.*

Proof. If verifying that a sequence of ABoxes is a repair is in P and verifying the entailment, and thus also the non-entailment, of a BTCQ is in PSPACE, the procedures ARNonEntailment, IARNonEntailment, and braveEntailment all run in NPSPACE=PSPACE. Moreover, coNPSPACE=PSPACE. □

Proposition 6.5 applies to DL-Lite$_\mathcal{R}$ and $\mathcal{EL}_\bot$ in all cases except for $\mathcal{EL}_\bot$ with rigid roles, for which BTCQ entailment under classical semantics is coNEXPTIME-hard [22].

Proposition 6.6. *BTCQ entailment from an $\mathcal{EL}_\bot$ TKB under AR, IAR and brave semantics is in* coNEXPTIME *w.r.t. combined complexity, even if $\mathsf{N_{RR}} \neq \emptyset$.*

Proof. For the AR and IAR semantics, we modify the procedures ARNonEntailment and IARNonEntailment described in Section 5 so that they also guess a certificate of the non-entailment of ϕ in the first step. Then, in the second step, the non-entailment of ϕ can be decided by simply verifying this certificate. The certificate can be checked in EXPTIME, since the non-entailment of ϕ can be decided in NEXPTIME.

For the brave semantics' upper bound, we give a NEXPTIME procedure to decide $\mathcal{K}, p \not\models_{\text{brave}} \phi$. For every subset $(\mathcal{A}'_i)_{0\leqslant i\leqslant n}$ of $(\mathcal{A}_i)_{0\leqslant i\leqslant n}$, guess either "not a repair" or a certificate of the non-entailment of ϕ from $\langle\mathcal{T}, (\mathcal{A}'_i)_{0\leqslant i\leqslant n}\rangle$ at time point p. Note that there are $2^{|(\mathcal{A}_i)_{0\leqslant i\leqslant n}|}$ such subsets. For every such subset, verify in EXPTIME whether it is indeed not a repair, or whether $\langle\mathcal{T}, (\mathcal{A}'_i)_{0\leqslant i\leqslant n}\rangle, p \not\models \phi$. □

The matching PSPACE and coNEXPTIME combined complexity lower bounds for $\mathcal{EL}_\bot$ and DL-Lite$_\mathcal{R}$ follow from the consistent case (cf. Section 2).

6.3. Data complexity for $\mathcal{EL}_\bot$ TKBs

We now prove the data complexity results, starting with $\mathcal{EL}_\bot$. We first consider the case without rigid predicates.

Proposition 6.7. *BTCQ entailment from an $\mathcal{EL}_\bot$ TKB with $\mathsf{N_{RC}} = \mathsf{N_{RR}} = \emptyset$ is*

- *coNP-complete w.r.t. data complexity under AR and IAR semantics, and*
- *NP-complete w.r.t. data complexity under brave semantics.*

Proof. The upper bounds follow from the procedures described in Section 5: since verifying that a sequence of ABoxes is a repair as well as verifying the non-entailment and entailment of a BTCQ take polynomial time w.r.t. data complexity, the procedures ARNonEntailment, IARNonEntailment, and braveEntailment run in NP w.r.t. data complexity. The lower bounds follow from the atemporal case. □

Next, we prove the complexity of BTCQ entailment with rigid predicates. The following proposition establishes the upper bounds for the case where both rigid concepts and rigid roles are allowed.

Proposition 6.8. *BTCQ entailment from an $\mathcal{EL}_\bot$ TKB with $\mathsf{N_{RC}} \neq \emptyset$ and $\mathsf{N_{RR}} \neq \emptyset$ is*

- *in coNP w.r.t. data complexity under AR and IAR semantics, and*

– *in* Σ_2^p *w.r.t. data complexity under brave semantics.*

Proof. For AR and IAR semantics, we modify the procedures described in Section 5 to also guess a certificate for the non-entailment of ϕ. This certificate can be checked in P, since the non-entailment of ϕ can be decided in NP. The upper bound for brave semantics is obtained using the procedure braveEntailment described in Section 5. $\square$

We show that these results are tight even if we only have rigid concepts.

Proposition 6.9. *BTCQ entailment from an* $\mathcal{EL}_\bot$ *TKB with* $\mathsf{N_{RC}} \neq \emptyset$ *is*

– *co*NP*-hard w.r.t. data complexity under AR and IAR semantics, and*
– Σ_2^p*-hard w.r.t. data complexity under brave semantics.*

Proof (Sketch). The lower bounds for AR and IAR semantics follow from the atemporal case. For brave semantics, we show that the complement of brave TCQ entailment is Π_2^p-hard by reduction from $\mathrm{QBF}_{2,\forall}$.

Let $\Phi = \forall x_1 \dots x_m \exists y_1 \dots y_r\, \varphi$ be a $\mathrm{QBF}_{2,\forall}$-formula, where $\varphi = \bigwedge_{i=0}^{h} \ell_i^0 \vee \ell_i^1 \vee \ell_i^2$ is a 3-CNF formula over the propositional variables $\{x_1, \dots, x_m, y_1, \dots, y_r\}$. Based on Φ, we define the TKB $\mathcal{K} = \langle \mathcal{T}, (\mathcal{A}_i)_{0 \leqslant i \leqslant 3h+2} \rangle$ and the TCQ ϕ as follows, where $\mathsf{N_{RC}} = \{\mathsf{T}\}$.

$$\begin{aligned}\mathcal{T} = \{&\exists \mathsf{Pos}.\mathsf{T} \sqsubseteq \mathsf{Sat}, \exists \mathsf{Neg}.\mathsf{F} \sqsubseteq \mathsf{Sat},\\ &\exists \mathsf{FromPos}.\mathsf{Sat} \sqsubseteq \mathsf{T}, \exists \mathsf{FromNeg}.\mathsf{Sat} \sqsubseteq \mathsf{F},\\ &\exists \mathsf{FromY}.\mathsf{Sat} \sqsubseteq \mathsf{T}, \mathsf{T} \sqcap \mathsf{F} \sqsubseteq \bot,\\ &\mathsf{T} \sqcap \exists \mathsf{ValY}.\mathsf{T} \sqsubseteq \bot\}\\ \phi =& \neg \Box^b(\mathsf{NotFirst}(c) \vee \mathsf{Sat}(c) \vee\\ &\bigcirc \mathsf{Sat}(c) \vee \bigcirc\bigcirc \mathsf{Sat}(c))\end{aligned}$$

For each clause $\ell_i^0 \vee \ell_i^1 \vee \ell_i^2$, we define the following three ABoxes $\mathcal{A}_{3i+k}$ $(0 \leqslant k \leqslant 2)$:

$$\begin{aligned}\mathcal{A}_{3i} =& \mathcal{B} \cup \mathcal{B}_{3i}\\ \mathcal{A}_{3i+k} =& \mathcal{B} \cup \mathcal{B}_{3i+k} \cup \{\mathsf{NotFirst}(c)\}, 1 \leqslant k \leqslant 2,\end{aligned}$$

where

$$\begin{aligned}\mathcal{B} =& \{\mathsf{T}(x_j), \mathsf{F}(x_j) \mid 1 \leqslant j \leqslant m\} \cup\\ &\{\mathsf{ValY}(y_j, \neg y_j), \mid 1 \leqslant j \leqslant r\}\end{aligned}$$

$$\begin{aligned}\mathcal{B}_{3i+k} =& \{\mathsf{Pos}(c, x_j), \mathsf{FromPos}(x_j, c)\} \text{ if } \ell_i^k = x_j\\ \mathcal{B}_{3i+k} =& \{\mathsf{Neg}(c, x_j), \mathsf{FromNeg}(x_j, c)\} \text{ if } \ell_i^k = \neg x_j\\ \mathcal{B}_{3i+k} =& \{\mathsf{FromY}(y_j, c)\} \text{ if } \ell_i^k = y_j\\ \mathcal{B}_{3i+k} =& \{\mathsf{FromY}(\neg y_j, c)\} \text{ if } \ell_i^k = \neg y_j.\end{aligned}$$

We show that Φ is valid iff $\mathcal{K}, 0 \not\models_{\text{brave}} \phi$.

Since T is rigid and disjoint from F, the repairs of $\mathcal{K}$ correspond to the valuations of the x_j.

Assume Φ is valid. We can then show that for every repair $(\mathcal{A}_i')_{0 \leqslant i \leqslant 3h+2}$ of $\mathcal{K}$, we can define a model $\mathcal{J} = (\mathcal{I}_i)_{i \geqslant 0}$ of $\langle \mathcal{T}, (\mathcal{A}_i')_{0 \leqslant i \leqslant 3h+2} \rangle$ that respects rigid predicates and such that $\mathcal{J}, 0 \models \Box^b(\mathsf{NotFirst}(c) \vee \mathsf{Sat}(c) \vee \bigcirc \mathsf{Sat}(c) \vee \bigcirc\bigcirc \mathsf{Sat}(c))$. Indeed, since Φ is valid, there exists a valuation ν_Y of the y_j that satisfies φ together with the valuation of the x_j corresponding to $(\mathcal{A}_i')_{0 \leqslant i \leqslant 3h+2}$. This valuation can be used to define $\mathcal{J}$ as follows.

– $y_j^{\mathcal{I}_i} \in \mathsf{T}^{\mathcal{I}_i}$ iff $\nu_Y(y_j) = \mathsf{true}$
– $\neg y_j^{\mathcal{I}_i} \in \mathsf{T}^{\mathcal{I}_i}$ iff $\nu_Y(y_j) = \mathsf{false}$
– if there exists some d such that $(d^{\mathcal{I}_i}, c^{\mathcal{I}_i}) \in \mathsf{FromY}^{\mathcal{I}_i}$ and $d^{\mathcal{I}_i} \in \mathsf{T}^{\mathcal{I}_i}$, then $c^{\mathcal{I}_i} \in \mathsf{Sat}^{\mathcal{I}_i}$.

It is then easy to see that at each time point $p \in [\![0, 3h+2]\!]$, either $\mathsf{NotFirst}(c)$ is true, or $p = 3i$ and $\mathsf{Sat}(c)$ is true at time point $3i + k$, where ℓ_i^k is the first satisfied literal of the clause $\ell_i^0 \vee \ell_i^1 \vee \ell_i^2$. Since $\mathcal{J}, 0 \not\models \phi$, then $\langle \mathcal{T}, (\mathcal{A}_i')_{0 \leqslant i \leqslant 3h+2} \rangle, 0 \not\models \phi$. Thus $\mathcal{K}, 0 \not\models_{\text{brave}} \phi$.

Now assume $\mathcal{K}, 0 \not\models_{\text{brave}} \phi$. We can then show that for every valuation ν_X of the x_j, there exists a valuation ν_Y of the y_j such that φ is satisfied. Indeed, there exists a model $\mathcal{J}$ of the repair corresponding to ν_X and of the TBox that respects rigid predicates and is such that $\mathcal{J}, 0 \not\models \phi$, i.e.,

$$\mathcal{J}, 0 \models \Box^b(\mathsf{NotFirst}(c) \vee \mathsf{Sat}(c) \vee \bigcirc \mathsf{Sat}(c) \vee \bigcirc\bigcirc \mathsf{Sat}(c)).$$

This model can be used to define ν_Y. For this, we set $\nu_Y(y_j) = \mathsf{true}$ if there exists k such that $\mathcal{J}, k \models \mathsf{T}(y_j)$, and $\nu_Y(y_j) = \mathsf{false}$ if there exists k such that $\mathcal{J}, k \models \mathsf{T}(\neg y_j)$. Since $\mathcal{J}, 0 \models \Box^b(\mathsf{NotFirst}(c) \vee \mathsf{Sat}(c) \vee \bigcirc \mathsf{Sat}(c) \vee \bigcirc\bigcirc \mathsf{Sat}(c))$, for every clause $\ell_i^0 \vee \ell_i^1 \vee \ell_i^2$, we have that $\mathcal{J}, 3i + k \models \mathsf{Sat}(c)$ for some $k \in [\![0, 2]\!]$, and we can show that ℓ_i^k is then evaluated to true. It follows that $\nu_X \cup \nu_Y$ satisfies every clause $\ell_i^0 \vee \ell_i^1 \vee \ell_i^2$. Hence, $\varphi[x_j \leftarrow \nu_X(x_j), y_j \leftarrow \nu_Y(y_j)]$ evaluates to true for every valuation ν_X, which implies that Φ is valid. $\square$

6.4. Data complexity for DL-Lite$_\mathcal{R}$ TKBs

It remains to show the data complexity results for DL-Lite$_\mathcal{R}$. We first consider the case where $\mathsf{N_{RC}} \neq \emptyset$. For AR and brave semantics, the upper bounds follow from the guess and check procedures described in Section 5 (for brave semantics, guess together with the repair a certificate that it entails ϕ at time point p). The lower bound for AR follows from the atemporal case, which establishes a tight coNP-bound even if $\mathsf{N_{RC}} = \mathsf{N_{RR}} = \emptyset$. In contrast, for brave semantics, BCQ entailment is tractable in the atemporal setting. However, we cannot directly extend this result to the temporal case. Indeed, the data complexity upper bound for brave CQ answering in DL-Lite$_\mathcal{R}$ relies on the fact that the size of the minimal sets of assertions that support the query is bounded by the query size. This is not true in the temporal setting, as can already be seen by the query $\phi = \Box^- A(a)$, whose entailment at time point p can depend on p assertions in the TKB. In fact, we show that in the presence of rigid concepts, brave BTCQ entailment becomes NP-hard.

Proposition 6.10. *If $\mathsf{N_{RC}} \neq \emptyset$, then brave BTCQ entailment from DL-Lite$_\mathcal{R}$ TKBs is* NP*-hard w.r.t. data complexity.*

Proof. We show NP-hardness of brave BTCQ entailment from DL-Lite$_\mathcal{R}$ TKBs by reduction from SAT. Let $\varphi = c_0 \wedge ... \wedge c_n$ be a CNF formula over variables $x_1, ..., x_m$. We define the following problem of BTCQ entailment under brave semantics, with two rigid concepts T and F. Let $\mathcal{K} = \langle \mathcal{T}, (\mathcal{A}_i)_{0 \leqslant i \leqslant n} \rangle$ be such that:

$$\begin{aligned} \mathcal{T} = \{&\exists \mathsf{Pos} \sqsubseteq \mathsf{Sat},\ \exists \mathsf{Neg} \sqsubseteq \mathsf{Sat},\\ &\exists \mathsf{Pos}^- \sqsubseteq \mathsf{T},\ \exists \mathsf{Neg}^- \sqsubseteq \mathsf{F},\ \mathsf{T} \sqsubseteq \neg \mathsf{F}\} \\ \mathcal{A}_i = \{&\mathsf{Pos}(a, x_j) \mid x_j \in c_i\} \cup \\ &\{\mathsf{Neg}(a, x_j) \mid \neg x_j \in c_i\} \ \text{ for } 0 \leqslant i \leqslant n \end{aligned}$$

Let $\phi = \Box^- \mathsf{Sat}(a)$. We show that φ is satisfiable iff $\mathcal{K}, n \models_{\text{brave}} \phi$. Indeed, since T and F are rigid, a repair $(\mathcal{A}'_i)_{0 \leqslant i \leqslant n}$ of $\mathcal{K}$ is such that each x_j has either only Pos or only Neg incoming edges in $(\mathcal{A}'_i)_{0 \leqslant i \leqslant n}$. We can thus define a valuation ν of the variables such that $\nu(x_j) = \mathsf{true}$ if $(\mathcal{A}'_i)_{0 \leqslant i \leqslant n}$ does not contain a timed assertion of the form $(\mathsf{Neg}(c, x_j), k)$, and $\nu(x_j) = \mathsf{false}$ otherwise. The clause c_i is satisfied by ν iff there exists x_j such that either $x_j \in c_i$ and $\nu(x_j) = \mathsf{true}$ or $\neg x_j \in c_i$ and $\nu(x_j) = \mathsf{false}$, that is, iff there exists x_j such that either $\mathsf{Pos}(a, x_j) \in \mathcal{A}'_i$ or $\mathsf{Neg}(a, x_j) \in \mathcal{A}'_i$, which holds exactly iff $\langle \mathcal{T}, (\mathcal{A}'_i)_{0 \leqslant i \leqslant n} \rangle, i \models \mathsf{Sat}(a)$. It follows that φ is satisfiable iff there exists a repair $(\mathcal{A}'_i)_{0 \leqslant i \leqslant n}$ of $\mathcal{K}$ that entails ϕ at time point n. $\square$

For the case $\mathsf{N_{RC}} = \mathsf{N_{RR}} = \emptyset$, we have an ALOGTIME lower bound from the classical semantics, and it is open whether the NP upper bound can be improved.

In contrast, for IAR semantics, we can give a tractable upper bound even if $\mathsf{N_{RR}} \neq \emptyset$. The reason is that, in DL-Lite$_\mathcal{R}$ TKBs, the size of a minimal inconsistent subset is at most two, as in the atemporal case.

Fact 6.11 ([28], Lemma 1)**.** Because of the DL-Lite$_\mathcal{R}$ syntax, the following holds: for every DL-Lite$_\mathcal{R}$ TBox $\mathcal{T}$, the size of a minimal $\mathcal{T}$-inconsistent set of (timed) assertions is at most two.

Therefore, we can always compute the intersection of all repairs in polynomial time.

Proposition 6.12. *BTCQ entailment from a DL-Lite$_\mathcal{R}$ TKB under IAR semantics is in* P *w.r.t. data complexity, even if $\mathsf{N_{RR}} \neq \emptyset$ and $\mathsf{N_{RC}} \neq \emptyset$.*

Proof. The size of the minimal $\mathcal{T}$-inconsistent subsets of $(\mathcal{A}_i)_{0 \leqslant i \leqslant n}$ is bounded by 2. We can thus skip the first step of the procedure $\mathsf{IAREntailment}$ described in Section 5 and compute the minimal inconsistent subsets in P by checking the consistency of every timed assertion and pair of timed assertions (with a quadratic number of consistency checks), and then verify the entailment of the query in P w.r.t. data complexity over the TKB from which they have been removed. $\square$

7. BTCQ entailment under classical semantics without negation in the query

This section completes the complexity picture for BTCQ entailment under the classical semantics by investigating the case where TCQs do not contain negation. We show that the absence of negation in the query induces a complexity drop in several cases. These results are based on a more general property: we show that for any DL $\mathcal{L}$, if $\mathcal{L}$ has the canonical model property for CQ answering over KBs, then $\mathcal{L}$ has also the canonical model property for TCQ answering over TKBs for TCQs without negation. We use the canonical model to prove that for the classical semantics, the complexity upper bounds of the atemporal case transfer to the temporal case.

We first show that BCQ entailment from a TKB $\mathcal{K}$ can be reduced to BCQ entailment from the KB $\widetilde{\mathcal{K}}$ defined Section 6, Figure 4. For this, we define a similar transformation for BCQs as we did for TKBs. Let $q = \exists\vec{y}.\psi(\vec{y})$ be a BCQ and $p \geqslant 0$ be a time point. Consider

$$\widetilde{q}_p = \mathsf{RenameNotRig}(q, p)$$

where $\mathsf{RenameNotRig}(q, p)$ replaces every non-rigid predicate X in q by X_p if $p \leqslant n$, and by X_{n+1} otherwise.

Lemma 7.1. *$\mathcal{K}, p \models q$ iff $\widetilde{\mathcal{K}} \models \widetilde{q}_p$.*

Proof. The following proof is written for the case $p \in [\![0, n+1]\!]$. For the case $p > n+1$, p is replaced by $n+1$ in the predicates names.

Assume that $\mathcal{K}, p \models q$, and let $\widetilde{\mathcal{I}}$ be a model of $\widetilde{\mathcal{K}}$. Let $\mathcal{J} = (\mathcal{I}_i)_{i\geqslant 0} = \mathsf{temp}(\widetilde{\mathcal{I}})$ be the corresponding model of $\mathcal{K}$ that respects rigid predicates, as defined in the proof for Lemma 6.2. For any BCQ ψ without existential variables, we denote by $\mathsf{atoms}(\psi)$ the set of (ground) atoms of ψ. Since $\mathcal{I}_p \models q$, there then exists a mapping π from the set of constants and variables that appear in ψ into Δ such that for every $X(\vec{d}) \in \mathsf{atoms}(\psi_\pi)$, where ψ_π is the BCQ obtained by replacing the terms of ψ by their image by π, we have $\vec{d} \in X^{\mathcal{I}_p}$. It follows that for every $X(\vec{d}) \in \mathsf{atoms}(\psi_\pi)$, if X is rigid then $\vec{d} \in X^{\widetilde{\mathcal{I}}}$, and otherwise $\vec{d} \in X_p^{\widetilde{\mathcal{I}}}$. Thus, $\widetilde{\mathcal{I}} \models \mathsf{RenameNotRig}(q, p)$, i.e., $\widetilde{\mathcal{I}} \models \widetilde{q}_p$. Hence $\widetilde{\mathcal{K}} \models \widetilde{q}_p$.

In the other direction, assume that $\widetilde{\mathcal{K}} \models \widetilde{q}_p$ and let $\mathcal{J} = (\mathcal{I}_i)_{i\geqslant 0}$ be a model of $\mathcal{K}$ that respects rigid predicates. Let $\widetilde{\mathcal{I}} = \mathsf{atemp}(\mathcal{J})$ be as defined in the proof for Lemma 6.2. Since $\widetilde{\mathcal{I}} \models \widetilde{q}_p$, there then exists a mapping π from the set of constants and variables that appear in ψ into Δ such that for every $X(\vec{d}) \in \mathsf{atoms}(\mathsf{RenameNotRig}(\psi_\pi, p))$, we have $\vec{d} \in X^{\widetilde{\mathcal{I}}}$. It follows that for every $X(\vec{d}) \in \mathsf{atoms}(\mathsf{RenameNotRig}(\psi_\pi, p))$ such that X is rigid, $\vec{d} \in X^{\mathcal{I}_p}$. Furthermore, we have $\vec{d} \in X^{\mathcal{I}_p}$ for every $X_p(\vec{d}) \in \mathsf{atoms}(\mathsf{RenameNotRig}(\psi_\pi, p))$ such that X is not rigid. Thus, $\mathcal{I}_p \models q$, and we obtain $\mathcal{K}, p \models q$. □

Moreover, the size of $\widetilde{q}_p$ is the same as q. Hence:

Lemma 7.2. *If BCQ entailment from an $\mathcal{L}$ KB is in* P *w.r.t. KB complexity and in* NP *w.r.t. combined complexity, then so is BCQ entailment from an $\mathcal{L}$ TKB.*

Proof. Since deciding whether $\widetilde{\mathcal{K}} \models \widetilde{q}_p$ is polynomial both in $|\widetilde{\mathcal{T}}|$ and in $|\widetilde{\mathcal{A}}|$, it is polynomial in $|\mathcal{T}|$ and $|(\mathcal{A}_i)_{0\leqslant i\leqslant n}|$. It follows that deciding whether $\mathcal{K}, p \models q$ is in P w.r.t. KB complexity.

Moreover, since deciding whether $\widetilde{\mathcal{K}} \models \widetilde{q}_p$ is in NP w.r.t. $|\widetilde{\mathcal{T}}|$, $|\widetilde{\mathcal{A}}|$ and $|\widetilde{q}_p|$, then verifying a certificate that $\widetilde{\mathcal{K}} \models \widetilde{q}_p$ can be done in polynomial time w.r.t. $|\widetilde{\mathcal{T}}|$, $|\widetilde{\mathcal{A}}|$ and $|\widetilde{q}_p|$, so in polynomial time w.r.t. $|\mathcal{T}|$, n, $|(\mathcal{A}_i)_{0\leqslant i\leqslant n}|$ and $|q|$. It follows that deciding whether $\mathcal{K}, p \models q$ is in NP w.r.t. combined complexity. □

We next define the notion of canonical model property for BCQ entailment and for entailment of BTCQ without negation.

Definition 7.3 (Canonical model property)**.** A DL $\mathcal{L}$ has the *canonical model property for BCQ entailment* iff for every $\mathcal{L}$ KB $\langle\mathcal{T}, \mathcal{A}\rangle$, there exists a model $\mathcal{I}_{\langle\mathcal{T},\mathcal{A}\rangle}$ such that for every BCQ q, $\langle\mathcal{T}, \mathcal{A}\rangle \models q$ iff $\mathcal{I}_{\langle\mathcal{T},\mathcal{A}\rangle} \models q$. We call $\mathcal{I}_{\langle\mathcal{T},\mathcal{A}\rangle}$ the canonical model of $\langle\mathcal{T}, \mathcal{A}\rangle$.

A DL $\mathcal{L}$ has the *canonical model property for entailment of BTCQ without negation* iff for any $\mathcal{L}$ TKB $\langle\mathcal{T}, (\mathcal{A}_i)_{0\leqslant i\leqslant n}\rangle$, there exists a model $\mathcal{J}_{\langle\mathcal{T},(\mathcal{A}_i)_{0\leqslant i\leqslant n}\rangle}$ such that for every BTCQ without negation ϕ and every time point p,

$$\langle\mathcal{T}, (\mathcal{A}_i)_{0\leqslant i\leqslant n}\rangle, p \models \phi \text{ iff } \mathcal{J}_{\langle\mathcal{T},(\mathcal{A}_i)_{0\leqslant i\leqslant n}\rangle}, p \models \phi.$$

We call the model $\mathcal{J}_{\langle\mathcal{T},(\mathcal{A}_i)_{0\leqslant i\leqslant n}\rangle}$ the canonical model of $\langle\mathcal{T}, (\mathcal{A}_i)_{0\leqslant i\leqslant n}\rangle$.

Note that it is justified to speak of *the* canonical model of a KB or TKB $\mathcal{K}$ because such a model can be homomorphically mapped into any other model of $\mathcal{K}$. Indeed, for every assertion α built over $\mathsf{N_I}$, $\mathsf{N_C}$ and $\mathsf{N_R}$, if α holds in the canonical model of $\mathcal{K}$ then it also holds in every model of $\mathcal{K}$.

The following theorem gives the relation between the canonical model property for BCQ entailment and for BTCQ entailment and shows why the presence or absence of negations in the query matters.

Theorem 7.4. *If $\mathcal{L}$ has the canonical model property for BCQ entailment, then $\mathcal{L}$ has also the canonical model property for the entailment of BTCQ without negation.*

Proof. Let $\widetilde{\mathcal{I}}_{\widetilde{\mathcal{K}}}$ be the canonical model of $\widetilde{\mathcal{K}}$ and $\mathcal{J}_{\mathcal{K}} = (\mathcal{I}_i)_{i\geqslant 0} = \mathsf{temp}(\widetilde{\mathcal{I}}_{\widetilde{\mathcal{K}}})$. We show that $\mathcal{J}_{\mathcal{K}}$ is the canonical model of $\mathcal{K}$ for BTCQs without negation, that is, for every BTCQ ϕ that does not contain any negation, $\mathcal{K}, p \models \phi$ iff $\mathcal{J}_{\mathcal{K}}, p \models \phi$.

Since $\mathcal{J}_\mathcal{K}$ is a model of $\mathcal{K}$ that respects rigid predicates, if $\mathcal{K}, p \models \phi$ then $\mathcal{J}_\mathcal{K}, p \models \phi$. For the other direction, we show by induction on the structure of ϕ that if $\mathcal{J}_\mathcal{K}, p \models \phi$, then $\mathcal{K}, p \models \phi$.

If $\phi = q$ is a BCQ, by Lemma 7.1, $\mathcal{K}, p \models q$ iff $\widetilde{\mathcal{K}} \models \widetilde{q}_p$, which is exactly the case iff $\widetilde{\mathcal{I}}_{\widetilde{\mathcal{K}}} \models \widetilde{q}_p$. By construction of $\mathcal{J}_\mathcal{K}$, it follows that $\mathcal{K}, p \models q$ iff $\mathcal{J}_\mathcal{K}, p \models q$.

Assume that for two BTCQs ϕ_1, ϕ_2 and any $p \geqslant 0$, $\mathcal{J}_\mathcal{K}, p \models \phi_i$ implies $\mathcal{K}, p \models \phi_i$ ($i \in \{1, 2\}$). We can show the following for BTCQs built from ϕ_1, ϕ_2.

- If $\mathcal{J}_\mathcal{K}, p \models \phi_1 \wedge \phi_2$ then $\mathcal{J}_\mathcal{K}, p \models \phi_1$ and $\mathcal{J}_\mathcal{K}, p \models \phi_2$. Hence by assumption, $\mathcal{K}, p \models \phi_1$ and $\mathcal{K}, p \models \phi_2$, thus $\mathcal{K}, p \models \phi_1 \wedge \phi_2$.
- If $\mathcal{J}_\mathcal{K}, p \models \phi_1 \vee \phi_2$ then $\mathcal{J}_\mathcal{K}, p \models \phi_1$ or $\mathcal{J}_\mathcal{K}, p \models \phi_2$. Hence by assumption, $\mathcal{K}, p \models \phi_1$ or $\mathcal{K}, p \models \phi_2$, thus $\mathcal{K}, p \models \phi_1 \vee \phi_2$.
- If $\mathcal{J}_\mathcal{K}, p \models \bigcirc\phi_1$ then $\mathcal{J}_\mathcal{K}, p + 1 \models \phi_1$. Hence by assumption, $\mathcal{K}, p + 1 \models \phi_1$, thus $\mathcal{K}, p \models \bigcirc\phi_1$.
- We can show similarly that if $\mathcal{J}_\mathcal{K}, p \models \bullet^b \phi_1$ then $\mathcal{K}, p \models \bullet^b \phi_1$, that if $\mathcal{J}_\mathcal{K}, p \models \bigcirc^- \phi_1$ then $\mathcal{K}, p \models \bigcirc^- \phi_1$, and that if $\mathcal{J}_\mathcal{K}, p \models \bullet^- \phi_1$ then $\mathcal{K}, p \models \bullet^- \phi_1$.
- If $\mathcal{J}_\mathcal{K}, p \models \Box\phi_1$ then for every $k \geqslant p$, $\mathcal{J}_\mathcal{K}, k \models \phi_1$. Hence by assumption, for every $k \geqslant p$, $\mathcal{K}, k \models \phi_1$, thus $\mathcal{K}, p \models \Box\phi_1$.
- We can show similarly that if $\mathcal{J}_\mathcal{K}, p \models \Box^b\phi_1$ then $\mathcal{K}, p \models \Box^b\phi_1$, and that if $\mathcal{J}_\mathcal{K}, p \models \Box^-\phi_1$ then $\mathcal{K}, p \models \Box^-\phi_1$.
- If $\mathcal{J}_\mathcal{K}, p \models \Diamond\phi_1$ then there exists $k \geqslant p$, $\mathcal{J}_\mathcal{K}, k \models \phi_1$. Hence by assumption, there exists $k \geqslant p$, $\mathcal{K}, k \models \phi_1$, thus $\mathcal{K}, p \models \Diamond\phi_1$.
- We can show similarly that if $\mathcal{J}_\mathcal{K}, p \models \Diamond^b\phi_1$ then $\mathcal{K}, p \models \Diamond^b\phi_1$, and that if $\mathcal{J}_\mathcal{K}, p \models \Diamond^-\phi_1$ then $\mathcal{K}, p \models \Diamond^-\phi_1$.
- If $\mathcal{J}_\mathcal{K}, p \models \phi_1 \mathsf{U}\, \phi_2$ then there exists $k \geqslant p$, $\mathcal{J}_\mathcal{K}, k \models \phi_2$ and for every j such that $p \leqslant j < k$, $\mathcal{J}_\mathcal{K}, j \models \phi_1$. Hence by assumption, there exists $k \geqslant p$, $\mathcal{K}, k \models \phi_2$ and for every j such that $p \leqslant j < k$, $\mathcal{K}, j \models \phi_1$, thus $\mathcal{K}, p \models \phi_1 \mathsf{U}\, \phi_2$.
- We can show similarly that if $\mathcal{J}_\mathcal{K}, p \models \phi_1 \mathsf{U}^b \phi_2$ then $\mathcal{K}, p \models \phi_1 \mathsf{U}^b \phi_2$, and that if $\mathcal{J}_\mathcal{K}, p \models \phi_1 \mathsf{S}\, \phi_2$ then $\mathcal{K}, p \models \phi_1 \mathsf{S}\, \phi_2$.

We conclude that for every BTCQ without negation ϕ, the following holds: if $\mathcal{J}_\mathcal{K}, p \models \phi$, then $\mathcal{K}, p \models \phi$. □

Remark 7.5. If ϕ contains some negation, the preceding induction does not work and $\mathcal{J}_\mathcal{K}$ is not a canonical model for TCQ answering: $\mathcal{J}_\mathcal{K}, p \models \neg\phi$ does not guarantee that $\mathcal{J}, p \not\models \phi$ for every model $\mathcal{J}$ that respects rigid predicates. For example, consider $\mathcal{T} = \emptyset$ and $\mathcal{A}_i = \emptyset$ for every $i \in [\![0, n]\!]$. We have $\mathcal{J}_\mathcal{K}, 0 \not\models \exists x.A(x)$, but we can easily construct a model $\mathcal{J}$ for $\langle \mathcal{T}, (\mathcal{A}_i)_{0 \leqslant i \leqslant n} \rangle$ such that $\mathcal{J}, 0 \models \exists x.A(x)$.

The following proposition is a direct consequence of the existence of a canonical model for entailment of BTCQ without negation.

Proposition 7.6. *If $\mathcal{L}$ has the canonical model property for BCQ entailment, for every $\mathcal{L}$ TKBs $\mathcal{K}$ and $\mathcal{K}'$, if $\mathcal{K}$ and $\mathcal{K}'$ coincide for BCQ entailment, then $\mathcal{K}$ and $\mathcal{K}'$ coincide for entailment of BTCQ without negation, i.e., if for every time point p and BCQ q, $\mathcal{K}, p \models q$ iff $\mathcal{K}', p \models q$, then for every time point p and BTCQ without negation ϕ, $\mathcal{K}, p \models \phi$ iff $\mathcal{K}', p \models \phi$.*

Proof. If for every time point p and BCQ q, $\mathcal{K}, p \models q$ iff $\mathcal{K}', p \models q$, we can then show by induction on the structure of ϕ that $\mathcal{K}, p \models \phi$ iff $\mathcal{K}', p \models \phi$. For $\phi = \exists\vec{y}.\psi(\vec{y})$, this holds by assumption. Assume that for two BTCQs ϕ_1, ϕ_2, $\mathcal{K}, p \models \phi_i$ iff $\mathcal{K}', p \models \phi_i$ ($i \in \{1, 2\}$). Then, since by Theorem 7.4 $\mathcal{L}$ has the canonical model property for entailment of BTCQ without negation, by applying the definitions of BTCQ satisfaction of Table 1 to the canonical models of $\mathcal{K}$ and $\mathcal{K}'$, we obtain the following about formulas composed of ϕ_1 and ϕ_2.

- $\mathcal{K}, p \models \phi_1 \wedge \phi_2$ iff $\mathcal{K}, p \models \phi_1$ and $\mathcal{K}, p \models \phi_2$, which is the case iff $\mathcal{K}', p \models \phi_1$ and $\mathcal{K}', p \models \phi_2$ by assumption, i.e., iff $\mathcal{K}', p \models \phi_1 \wedge \phi_2$.
- $\mathcal{K}, p \models \phi_1 \vee \phi_2$ iff $\mathcal{K}, p \models \phi_1$ or $\mathcal{K}, p \models \phi_2$, which is the case iff $\mathcal{K}', p \models \phi_1$ or $\mathcal{K}', p \models \phi_2$ by assumption, i.e., iff $\mathcal{K}', p \models \phi_1 \vee \phi_2$.
- $\mathcal{K}, p \models \bigcirc\phi_1$ iff $\mathcal{K}, p + 1 \models \phi_1$, which is the case iff $\mathcal{K}', p+1 \models \phi_1$ by assumption, i.e., iff $\mathcal{K}', p \models \bigcirc\phi_1$.
- We show in the same way that $\mathcal{K}, p \models \bullet^b \phi_1$ iff $\mathcal{K}', p \models \bullet^b \phi_1$, that $\mathcal{K}, p \models \bigcirc^- \phi_1$ iff $\mathcal{K}', p \models \bigcirc^- \phi_1$, and that $\mathcal{K}, p \models \bullet^- \phi_1$ iff $\mathcal{K}', p \models \bullet^- \phi_1$.
- $\mathcal{K}, p \models \Box\phi_1$ iff for every $k, k \geqslant p$, $\mathcal{K}, k \models \phi_1$, which is the case iff for every k, $k \geqslant p$, $\mathcal{K}', k \models \phi_1$ by assumption, i.e., iff $\mathcal{K}', p \models \Box\phi_1$.
- We show in the same way that $\mathcal{K}, p \models \Box^b\phi_1$ iff $\mathcal{K}', p \models \Box^b\phi_1$, and that $\mathcal{K}, p \models \Box^-\phi_1$ iff $\mathcal{K}', p \models \Box^-\phi_1$.
- $\mathcal{K}, p \models \Diamond\phi_1$ iff there exists k, $k \geqslant p$, $\mathcal{K}, k \models \phi_1$, which is the case iff there exists $k, k \geqslant p, \mathcal{K}', k \models \phi_1$ by assumption, i.e., iff $\mathcal{K}', p \models \Diamond\phi_1$.
- We show in the same way that $\mathcal{K}, p \models \Diamond^b\phi_1$ iff $\mathcal{K}', p \models \Diamond^b\phi_1$, and that $\mathcal{K}, p \models \Diamond^-\phi_1$ iff $\mathcal{K}', p \models \Diamond^-\phi_1$.

– $\mathcal{K}, p \models \phi_1 \mathsf{U}\, \phi_2$ iff there exists k, $k \geqslant p$, $\mathcal{K}, k \models \phi_2$ and for every j, $p \leqslant j < k$, $\mathcal{K}, j \models \phi_1$, which is the case iff there exists k, $k \geqslant p$, $\mathcal{K}', k \models \phi_2$ and for every j, $p \leqslant j < k$, $\mathcal{K}', j \models \phi_1$ by assumption, i.e., iff $\mathcal{K}', p \models \phi_1 \mathsf{U}\, \phi_2$.
– We show in the same way that $\mathcal{K}, p \models \phi_1 \mathsf{U}^b\, \phi_2$ iff $\mathcal{K}', p \models \phi_1 \mathsf{U}^b\, \phi_2$, and that $\mathcal{K}, p \models \phi_1 \mathsf{S}\, \phi_2$ iff $\mathcal{K}', p \models \phi_1 \mathsf{S}\, \phi_2$.

We conclude that for every BTCQ without negation ϕ and time point p, $\mathcal{K}, p \models \phi$ iff $\mathcal{K}', p \models \phi$. □

We now prove a central proposition for TCQ answering over TKBs in DLs that have the canonical model property for entailment of BTCQs without negation. It amounts to reducing the entailment of BTCQs with unbounded future operators to the entailment of BTCQs with only bounded future operators. These can then be answered by considering only a finite number of time points.

Let $\mathcal{K}^*$ be the following TKB:

$$\mathcal{K}^* = \langle \mathcal{T}, (\mathcal{A}_i)_{0 \leqslant i \leqslant n} \cup (\mathcal{A}_{n+1}) \rangle \text{ where}$$
$$\mathcal{A}_{n+1} = \{A(a) \mid A \in \mathsf{N_{RC}}, A(a) \in \widetilde{\mathcal{A}}\} \cup \{R(a,b) \mid R \in \mathsf{N_{RR}}, R(a,b) \in \widetilde{\mathcal{A}}\}$$

Proposition 7.7. *If $\mathcal{L}$ has the canonical model property for BCQ entailment, the relations in Table 2 hold for any $\mathcal{L}$ TKB $\mathcal{K}$.*

Proof. It is easy to see that by construction $\mathcal{J}_{\mathcal{K}^*} = \mathcal{J}_{\mathcal{K}}$. Hence $\mathcal{K}, p \models \phi$ iff $\mathcal{K}^*, p \models \phi$ for every BTCQ without negation ϕ.

All relations in Table 2 but those for the operators $\Box$, $\Diamond$ and U are straightforwardly obtained by applying the definitions of BTCQ satisfaction of Table 1 to this canonical model. To show the three remaining relations, we rely on the fact that $\mathcal{J}_{\mathcal{K}^*}$ is such that for every $i > n$, $\mathcal{I}_i = \mathcal{I}_{n+1}$ and there is no past operators nested under unbounded future operators by definition of TCQs. Indeed, if a BTCQ ϕ_1 does not contain any past operators and $i > n$, $\mathcal{J}_{\mathcal{K}^*}, i \models \phi_1$ iff $\mathcal{J}_{\mathcal{K}^*}, n+1 \models \phi_1$. Hence we can show the following, from which the relations of Table 2 follow straightforwardly.

– $\mathcal{K}^*, p \models \Box\phi_1$ iff for every $k \geqslant p$, $\mathcal{K}^*, k \models \phi_1$. Hence $\mathcal{K}^*, p \models \Box\phi_1$ iff $\forall k$, $p \leqslant k \leqslant n$, $\mathcal{K}^*, k \models \phi_1$ and $\mathcal{K}^*, n+1 \models \phi_1$.

Table 2

Entailment under classical semantics for DLs with the canonical model property for BCQ entailment

ϕ	$\mathcal{K}, p \models \phi$ iff $\mathcal{K}^*, p \models \phi$ iff
$\exists \vec{y}.\psi(\vec{y})$	$\mathcal{K}^*, p \models \exists \vec{y}.\psi(\vec{y})$
$\phi_1 \wedge \phi_2$	$\mathcal{K}^*, p \models \phi_1$ and $\mathcal{K}^*, p \models \phi_2$
$\phi_1 \vee \phi_2$	$\mathcal{K}^*, p \models \phi_1$ or $\mathcal{K}^*, p \models \phi_2$
$\bigcirc\phi_1$	$\mathcal{K}^*, p+1 \models \phi_1$
$\bullet^b\, \phi_1$	$p < n$ implies $\mathcal{K}^*, p+1 \models \phi_1$
$\bigcirc^-\phi_1$	$p > 0$ and $\mathcal{K}^*, p-1 \models \phi_1$
$\bullet^-\phi_1$	$p > 0$ implies $\mathcal{K}^*, p-1 \models \phi_1$
$\Box\phi_1$	$\forall k$, $p \leqslant k \leqslant n+1$, $\mathcal{K}^*, k \models \phi_1$
$\Box^b\phi_1$	$\forall k$, $p \leqslant k \leqslant n$, $\mathcal{K}^*, k \models \phi_1$
$\Box^-\phi_1$	$\forall k$, $0 \leqslant k \leqslant p$, $\mathcal{K}^*, k \models \phi_1$
$\Diamond\phi_1$	$\exists k$, $p \leqslant k \leqslant n+1$, $\mathcal{K}^*, k \models \phi_1$
$\Diamond^b\phi_1$	$\exists k$, $p \leqslant k \leqslant n$, $\mathcal{K}^*, k \models \phi_1$
$\Diamond^-\phi_1$	$\exists k$, $0 \leqslant k \leqslant p$, $\mathcal{K}^*, k \models \phi_1$
$\phi_1 \mathsf{U}\, \phi_2$	$\exists k$, $p \leqslant k \leqslant n+1$, $\mathcal{K}^*, k \models \phi_2$ and $\forall j$, $p \leqslant j < k$, $\mathcal{K}^*, j \models \phi_1$
$\phi_1 \mathsf{U}^b\, \phi_2$	$\exists k$, $p \leqslant k \leqslant n$, $\mathcal{K}^*, k \models \phi_2$ and $\forall j$, $p \leqslant j < k$, $\mathcal{K}^*, j \models \phi_1$
$\phi_1 \mathsf{S}\, \phi_2$	$\exists k$, $0 \leqslant k \leqslant p$, $\mathcal{K}^*, k \models \phi_2$ and $\forall j$, $k < j \leqslant p$, $\mathcal{K}^*, j \models \phi_1$

– $\mathcal{K}^*, p \models \Diamond\phi_1$ iff there exists $k \geqslant p$, $\mathcal{K}^*, k \models \phi_1$. Hence $\mathcal{K}^*, p \models \Diamond\phi_1$ iff $\exists k$, $p \leqslant k \leqslant n$, $\mathcal{K}^*, k \models \phi_1$ or $\mathcal{K}^*, n+1 \models \phi_1$.
– $\mathcal{K}^*, p \models \phi_1 \mathsf{U}\, \phi_2$ iff $\exists k \geqslant p$, $\mathcal{K}^*, k \models \phi_2$ and $\forall j$, $p \leqslant j < k, \mathcal{K}^*, j \models \phi_1$. Hence $\mathcal{K}^*, p \models \phi_1 \mathsf{U}\, \phi_2$ iff $\exists k$, $p \leqslant k \leqslant n$, $\mathcal{K}^*, k \models \phi_2$ and $\forall j$, $p \leqslant j < k, \mathcal{K}^*, j \models \phi_1$, or $\mathcal{K}^*, n+1 \models \phi_2$ and $\forall j$, $p \leqslant j < n+1, \mathcal{K}^*, j \models \phi_1$.

□

In the next theorem, we transfer complexity upper bounds from the atemporal case to the temporal case (even with rigid predicates) for queries without negation and DLs that have the canonical model property for BCQ entailment. We consider DLs for which BCQ entailment is in P w.r.t. KB complexity and in NP w.r.t. combined complexity, such as DL-Lite$_{\mathcal{R}}$ and $\mathcal{EL}_{\bot}$.

Theorem 7.8. *If $\mathcal{L}$ has the canonical model property for BCQ entailment and is such that BCQ entailment from KBs is in* P *w.r.t. KB complexity and in* NP *w.r.t. combined complexity, then the entailment of BTCQs without negation from $\mathcal{L}$ TKBs is in* P *w.r.t. KB complexity and in* NP *w.r.t. combined complexity.*

Proof. By Lemma 7.2, it is possible to decide whether $\mathcal{K}^*, p \models q$ in P w.r.t. KB complexity for any BCQ q. Based on this, we can show by induction on the structure of ϕ that $\mathcal{K}^*, p \models \phi$ can be decided in P w.r.t. KB complexity. Assume that for two BTCQs ϕ_1, ϕ_2 and any $p \geqslant 0$, it is possible to decide in P whether

$\mathcal{K}^*, p \models \phi_i$. Using the relations in Table 2, we can prove the following.

- $\mathcal{K}^*, p \models \phi_1 \wedge \phi_2$ iff $\mathcal{K}^*, p \models \phi_1$ and $\mathcal{K}^*, p \models \phi_2$, so deciding whether $\mathcal{K}^*, p \models \phi_1 \wedge \phi_2$ can be done in P by checking that $\mathcal{K}^*, p \models \phi_1$ and $\mathcal{K}^*, p \models \phi_2$.
- $\mathcal{K}^*, p \models \phi_1 \vee \phi_2$ iff $\mathcal{K}^*, p \models \phi_1$ or $\mathcal{K}^*, p \models \phi_2$, so deciding whether $\mathcal{K}^*, p \models \phi_1 \vee \phi_2$ can be done in P by deciding whether $\mathcal{K}^*, p \models \phi_1$ and $\mathcal{K}^*, p \models \phi_2$ and checking that at least one is true.
- $\mathcal{K}^*, p \models \bigcirc\phi_1$ iff $\mathcal{K}^*, p+1 \models \phi_1$, so deciding whether $\mathcal{K}^*, p \models \bigcirc\phi_1$ can be done in P by checking whether $\mathcal{K}^*, p+1 \models \phi_1$.
- $\mathcal{K}^*, p \models \bullet^b \phi_1$ iff $p < n$ implies $\mathcal{K}^*, p+1 \models \phi_1$, so deciding whether $\mathcal{K}^*, p \models \bullet^b \phi_1$ can be done in P by checking whether $p \geqslant n$ or $\mathcal{K}^*, p+1 \models \phi_1$.
- We show in the same way that we can decide in P whether $\mathcal{K}^*, p \models \bigcirc^- \phi_1$ and whether $\mathcal{K}^*, p \models \bullet^- \phi_1$.
- $\mathcal{K}^*, p \models \Box\phi_1$ iff for every k, $p \leqslant k \leqslant n+1$, $\mathcal{K}^*, k \models \phi_1$, so deciding whether $\mathcal{K}^*, p \models \Box\phi_1$ can be done in P by checking for each $p \leqslant k \leqslant n+1$ that $\mathcal{K}^*, k \models \phi_1$.
- We show in the same way that we can decide in P whether $\mathcal{K}^*, p \models \Box^b\phi_1$ and whether $\mathcal{K}^*, p \models \Box^-\phi_1$.
- $\mathcal{K}^*, p \models \Diamond\phi_1$ iff there exists k, $p \leqslant k \leqslant n+1$, $\mathcal{K}^*, k \models \phi_1$, so deciding whether $\mathcal{K}^*, p \models \Diamond\phi_1$ can be done in P by deciding for each $p \leqslant k \leqslant n+1$ whether $\mathcal{K}^*, k \models \phi_1$, and checking that it is true for at least one k.
- We show in the same way that we can decide in P whether $\mathcal{K}^*, p \models \Diamond^b\phi_1$ and whether $\mathcal{K}^*, p \models \Diamond^-\phi_1$.
- $\mathcal{K}^*, p \models \phi_1 \mathsf{U} \phi_2$ iff there exists k, $p \leqslant k \leqslant n+1$, $\mathcal{K}^*, k \models \phi_2$, and for every j, $p \leqslant j < k$, $\mathcal{K}^*, j \models \phi_1$, so deciding whether $\mathcal{K}^*, p \models \phi_1 \mathsf{U} \phi_2$ can be done in P by deciding for each $p \leqslant k \leqslant n+1$ whether $\mathcal{K}^*, k \models \phi_1$ and whether $\mathcal{K}^*, k \models \phi_2$, and checking that the condition holds.
- We show in the same way that we can decide in P whether $\mathcal{K}^*, p \models \phi_1 \mathsf{U}^b \phi_2$ and whether $\mathcal{K}^*, p \models \phi_1 \mathsf{S} \phi_2$.

The number of subqueries in ϕ is linear w.r.t. the size of ϕ, and independent from the TKB size. It follows that the total number of polynomial checks is also polynomially bounded. Therefore, we obtain that for every BTCQ ϕ without negation, $\mathcal{K}^*, p \models \phi$ can be decided in P w.r.t. the size of $\mathcal{K}^*$. Since $\mathcal{K}, p \models \phi$ iff $\mathcal{K}^*, p \models \phi$ and the size of $\mathcal{K}^*$ is polynomial in the size of $\mathcal{K}$, deciding whether $\mathcal{K}, p \models \phi$ is in P w.r.t. KB complexity.

For the NP membership of entailment of BTCQs without negation w.r.t. combined complexity, we describe how to guess a certificate that $\mathcal{K}, p \models \phi$ that can be checked in P. This certificate consists of:

- a sequence of functions $(\nu_i)_{0 \leqslant i \leqslant n+1}$ that associate to each BCQ q of ϕ true or false, and
- for each BCQ q of ϕ and time point $i \in [\![0, n+1]\!]$ such that $\nu_i(q) = \mathsf{true}$: a certificate that $\widetilde{q}_i = \mathsf{RenameNotRig}(q, i)$ is entailed from $\widetilde{\mathcal{K}}$.

There are polynomially many pairs of a time point and a BCQ, and the certificate that $\widetilde{q}_i$ is entailed from $\widetilde{\mathcal{K}}$ can be checked in polynomial time, since BCQ entailment is in NP. Moreover, we can show that, since ϕ contains neither negations nor past operators nested under unbounded future operators, deciding whether the propositional abstraction of ϕ is satisfied by the sequence of truth assignments that assign the propositional abstraction of q to $\nu_i(q)$ for every $i \in [\![0, n+1]\!]$ and to $\nu_{n+1}(q)$ for every $i > n+1$ can be done in polynomial time w.r.t. the size of the query and the length of the sequence of ABoxes. Indeed, identify ϕ and the BCQs in it with their propositional abstractions, and denote by $w = w^0 w^1 ... w^n w^{n+1} ...$ the trace over $2^{BCQ(\phi)}$ (where $BCQ(\phi)$ is the set of BCQs of ϕ), such that $w^i = \{q \mid \nu_i(q) = \mathsf{true}\}$ for $i \leqslant n+1$, $w^i = w^{n+1}$ for $i > n+1$. Since $w^i = w^{n+1}$, for $i > n+1$, we can show similar relations as those in Table 2 for the entailment of LTL formulas without past operators nested under unbounded future operators or negations from w. We can then use a similar induction as we did when we proved the data complexity to show that $w, p \models \phi$ can be decided by checking which queries are in w^i. For this, the number of queries to be tested is polynomial in n and the size of ϕ. $\square$

As a consequence of Theorems 7.4 and 7.8, and since $\mathcal{EL}_\bot$ and DL-Lite$_\mathcal{R}$ have the canonical model property for BCQ entailment (cf. [11] for $\mathcal{EL}_\bot$, and [43] for DL-Lite$_\mathcal{R}$), we obtain the following theorem.

Theorem 7.9. *For DL-Lite$_\mathcal{R}$ and $\mathcal{EL}_\bot$, entailment of BTCQs without negation is in* P *w.r.t. KB complexity and in* NP *w.r.t. combined complexity, even if* $\mathsf{N_{RR}} \neq \emptyset$.

Besides these results for DL-Lite$_\mathcal{R}$ and $\mathcal{EL}_\bot$, the Theorems 7.4 and 7.8 hold for all Horn-DLs satisfying the complexity constraints in the precondition of Theorem 7.8. For instance, this holds for DL-Lite$^{\mathcal{N}}_{horn}$ [44].

8. Complexity of inconsistency-tolerant BTCQ entailment without negation in the query

The following proposition gives general complexity upper bounds for BTCQ entailment under the AR, IAR and brave semantics. By Theorem 7.9, they hold in particular for $\mathcal{L} = \mathcal{EL}_\bot$ and $\mathcal{L} =$ DL-Lite$_\mathcal{R}$ when negations are not allowed in the TCQs.

Proposition 8.1. *If $\mathcal{L}$ is such that consistency checking of a $\mathcal{L}$ TKB is in* P *and BTCQ entailment from a $\mathcal{L}$ TKB is in* P *w.r.t. data complexity and in* NP *w.r.t. combined complexity, then BTCQ entailment from a $\mathcal{L}$ TKB*

- *under AR semantics is in co*NP *w.r.t. data complexity and in Π_2^p w.r.t. combined complexity;*
- *under IAR semantics is in co*NP *w.r.t. data complexity and in $\Delta_2^p[O(log\ n)]$ w.r.t. combined complexity;*
- *under brave semantics is in* NP *w.r.t. data complexity and in* NP *w.r.t. combined complexity.*

Proof. The data complexities follow from the procedures described in Section 5: since verifying that a sequence of ABoxes is a repair as well as non-entailment and entailment of a BTCQ can be decided in polynomial time, ARNonEntailment, IARNonEntailment and braveEntailment take non-determistic polynomial time.

For the combined complexity of brave BTCQ entailment, a certificate that $\langle \mathcal{T}, (\mathcal{A}'_i)_{0 \leqslant i \leqslant n} \rangle, p \models \phi$ can be guessed together with $(\mathcal{A}'_i)_{0 \leqslant i \leqslant n}$, and verified in P.

For the combined complexity of IAR, we use the procedure IAREntailment of Section 5 with an NP oracle to decide the existence of a minimal inconsistent subset of size at least k in the first step, and to decide the entailment of ϕ in the last step. □

For $\mathcal{EL}_\bot$, matching lower bounds for all semantics come from the atemporal case [27, 34].

For DL-Lite$_\mathcal{R}$, we can obtain matching lower bounds from the atemporal case for the combined complexity of all semantics as well as for the data complexity of the AR semantics [28, 32]. Moreover, the proof of Proposition 6.10 does not use negation in the query, and therefore the data complexity lower bound for brave semantics with rigid predicates applies also in this case. Regarding IAR semantics, entailment of BTCQs with negations under IAR semantics is already in P (see Figure 3), so this better upper bound applies. Finally, we show that for brave semantics and DL-Lite$_\mathcal{R}$, in the case where there are no rigid predicates, we can improve the NP upper bound of Figure 3 to a P bound.

We describe a method for brave entailment of BTCQ without negation when $\mathsf{N_{RC}} = \mathsf{N_{RR}} = \emptyset$ that proceeds by type elimination over a set of tuples built from the query and that represent the TCQs that are entailed at each time point. First, we define the structure on which the method operates. We consider the set $BCQ(\phi)$ of *leaves* of ϕ, that is, the set of all BCQs in ϕ, and the set $F(\phi)$ of *subformulas* of ϕ. In what follows, we identify the BCQs of $BCQ(\phi)$ and the BTCQs of $F(\phi)$ with their *propositional abstractions*: if we write that a KB or a TKB entails some elements of $BCQ(\phi)$ or $F(\phi)$, we consider them as BCQs or BTCQs, and if we write that some elements of $BCQ(\phi)$ or $F(\phi)$ entail others, we consider the elements of $BCQ(\phi)$ as propositional variables and those of $F(\phi)$ as propositional LTL formulas built over these variables.

Definition 8.2. A *brave-justification structure* J for the BTCQ without negation ϕ in the TKB $\mathcal{K}$ is a set of tuples of the form $(i, L_{\text{now}}, F_{\text{now}}, F_{\text{prev}}, F_{\text{next}})$, where $0 \leqslant i \leqslant n$, $L_{\text{now}} \subseteq BCQ(\phi)$, $F_{\text{now}} \subseteq F(\phi)$, $F_{\text{prev}} \subseteq F(\phi)$, and $F_{\text{next}} \subseteq F(\phi)$.

Note that the size of a brave-justification structure for ϕ in $\mathcal{K} = \langle \mathcal{T}, (\mathcal{A}_i)_{0 \leqslant i \leqslant n} \rangle$ is linearly bounded in n and independent of the size of the ABoxes. A tuple $(i, L_{\text{now}}, F_{\text{now}}, F_{\text{prev}}, F_{\text{next}})$ is *justified in* J iff it fulfils all of the following conditions.

1. $\langle \mathcal{T}, \mathcal{A}_i \rangle \models_{\text{brave}} \bigwedge_{q \in L_{\text{now}}} q$.
2. If $i > 0$, there exists $(i-1, L'_{\text{now}}, F'_{\text{now}}, F'_{\text{prev}}, F'_{\text{next}}) \in J$ such that $F_{\text{prev}} = F'_{\text{now}}$ and $F_{\text{now}} = F'_{\text{next}}$.
3. If $i < n$, there exists $(i+1, L'_{\text{now}}, F'_{\text{now}}, F'_{\text{prev}}, F'_{\text{next}}) \in J$ such that $F_{\text{next}} = F'_{\text{now}}$ and $F_{\text{now}} = F'_{\text{prev}}$.
4. For every $\psi \in BCQ(\phi)$, if $F_{\text{now}} \models \psi$, then $\psi \in L_{\text{now}}$.
5. For every $\psi \in F(\phi)$, if $F_{\text{now}} \models \psi$, then $\psi \in F_{\text{now}}$.
6. For every $\psi \in F(\phi)$, if $\bigwedge_{q \in L_{\text{now}}} q \wedge \ocircle^- (\bigwedge_{\chi \in F_{\text{prev}}} \chi) \wedge \ocircle(\bigwedge_{\chi \in F_{\text{next}}} \chi) \models \psi$, then $\psi \in F_{\text{now}}$.
7. For every $\psi, \psi' \in F(\phi)$:
   if $\psi \vee \psi' \in F_{\text{now}}$, then either $\psi \in F_{\text{now}}$ or $\psi' \in F_{\text{now}}$,
   if $\Diamond\psi \in F_{\text{now}}$, then either $\psi \in F_{\text{now}}$ or $\Diamond\psi \in F_{\text{next}}$,
   if $\Diamond^b\psi \in F_{\text{now}}$, then either $\psi \in F_{\text{now}}$ or $\Diamond^b\psi \in F_{\text{next}}$,
   if $\Diamond^-\psi \in F_{\text{now}}$, then either $\psi \in F_{\text{now}}$ or $\Diamond^-\psi \in F_{\text{prev}}$,
   if $\psi' \mathsf{U}\, \psi \in F_{\text{now}}$, then either $\psi \in F_{\text{now}}$ or $\psi' \in F_{\text{now}}$ and $\psi' \mathsf{U}\, \psi \in F_{\text{next}}$,
   if $\psi' \mathsf{U}^b\, \psi \in F_{\text{now}}$, then either $\psi \in F_{\text{now}}$ or $\psi' \in F_{\text{now}}$ and $\psi' \mathsf{U}^b\, \psi \in F_{\text{next}}$,
   if $\psi' \mathsf{S}\, \psi \in F_{\text{now}}$, then either $\psi \in F_{\text{now}}$ or $\psi' \in F_{\text{now}}$ and $\psi' \mathsf{S}\, \psi \in F_{\text{prev}}$, and
   if ψ is of the form $\Box\varphi$, then either $\psi \notin F_{\text{now}}$ or $\langle \emptyset, (\emptyset)_{0 \leqslant i \leqslant n} \rangle, n+1 \models \Box\varphi$.

8. If $i = n$, then
for all $\psi \in F(\phi)$ that are of the form $\bullet^b \varphi$, $\psi \in F_{\text{now}}$,
for all $\psi \in F(\phi)$ that are of the form $\bigcirc\varphi$ and such that $\langle \emptyset, (\emptyset)_{0\leqslant i\leqslant n}\rangle, n+1 \not\models \Box\varphi$, we have $\psi \notin F_{\text{now}}$,
for all $\psi \in F(\phi)$ that are of the form $\Diamond^b\varphi, \Box^b\varphi$, or $\varphi' \mathsf{U}^b \varphi$, $\psi \in F_{\text{now}}$ iff $\varphi \in F_{\text{now}}$,
for all $\psi \in F(\phi)$ that of the form $\Diamond\varphi, \varphi'\mathsf{U}^b\varphi$ and such that $\langle \emptyset, (\emptyset)_{0\leqslant i\leqslant n}\rangle, n+1 \not\models \Box\varphi$, $\psi \in F_{\text{now}}$ iff $\varphi \in F_{\text{now}}$.
9. If $i = 0$, then
for all $\psi \in F(\phi)$ that are of the form $\bullet^-\varphi$, $\psi \in F_{\text{now}}$,
for all $\psi \in F(\phi)$ that are of the form $\bigcirc^-\varphi$, $\psi \notin F_{\text{now}}$, and
for all $\psi \in F(\phi)$ that are of the form $\Diamond^-\varphi, \Box^-\varphi, \varphi'\mathsf{S}\varphi$, $\psi \in F_{\text{now}}$ iff $\varphi \in F_{\text{now}}$.

We give the intuition behind the elements of the tuples that fulfil these conditions. The first element i is the time point we are considering, L_{now} is a set of BCQs whose conjunction is entailed under brave semantics by $\langle \mathcal{T}, \mathcal{A}_i\rangle$ (Condition 1), and F_{now} is the set of formulas that can be entailed together with L_{now}, depending on what is entailed in the previous and next time points, this information being stored in F_{prev} and F_{next} respectively (Condition 6). Conditions 2 and 3 ensure that there is a sequence of tuples representing every time point from 0 to n such that this information is coherent between consecutive tuples. Condition 4 expresses that L_{now} is exactly the set of BCQs contained in F_{now} and Condition 5 that F_{now} is maximal in the sense that it contains its consequences. Condition 7 enforces that F_{now}, F_{prev} and F_{next} respect the semantics of LTL operators and Conditions 8 and 9 enforce this semantics at the ends of the finite sequence. (Note that we use here the fact that past operators cannot be nested below unbounded future operators, and that no BCQ can be entailed under brave semantics after time point n because there are no rigid predicates.)

A brave-justification structure J is *correct* if every tuple is justified, and ϕ is *justified at time point p by J* if there is $(p, L_{\text{now}}, F_{\text{now}}, F_{\text{prev}}, F_{\text{next}}) \in J$ such that $\phi \in F_{\text{now}}$. We show that ϕ is entailed from $\mathcal{K}$ at time point p under brave semantics iff there is a correct brave-justification structure for ϕ in $\mathcal{K}$ that justifies ϕ at time point p. The main idea is to link the tuples of a sequence $((i, L_{\text{now}}, F_{\text{now}}, F_{\text{prev}}, F_{\text{next}}))_{0\leqslant i\leqslant n}$ to a consistent TKB $\mathcal{K}' = \langle \mathcal{T}, (\mathcal{C}_i)_{0\leqslant i\leqslant n}\rangle$ such that for every i, $\mathcal{C}_i \subseteq \mathcal{A}_i$ and $\langle \mathcal{T}, \mathcal{C}_i\rangle \models \bigwedge_{q\in L_{\text{now}}} q$. We show in the appendix that there is such a $\mathcal{K}'$ such that $\mathcal{K}', p \models \phi$ iff there is such a sequence of tuples that is a correct brave-justification structure for ϕ in $\mathcal{K}$ and justifies ϕ at time point p.

The data complexity of brave entailment of BTCQ without negation when there are no rigid predicates follows from the characterization of brave BTCQ entailment with brave-justification structures.

Proposition 8.3. *For DL-Lite$_{\mathcal{R}}$, if $\mathsf{N_{RC}} = \mathsf{N_{RR}} = \emptyset$, then entailment of BTCQs without negation under brave semantics is in* P *w.r.t. data complexity.*

Proof. We describe a polynomial procedure that decides the existence of a brave-justification structure for ϕ in $\mathcal{K}$ that justifies ϕ at time point p. We start with a brave-justification structure J for ϕ in $\mathcal{K}$ that contains all possible tuples. We then remove the unjustified tuples as follows: (i) remove every tuple that does not satisfy Conditions 1, 4, 5, 6, 7, 8 or 9, and (ii) repeat the following steps until a fix-point has been reached: iterate over the tuples from time point 0 to n, eliminating those which do not satisfy Condition 3, and then iterate from n to 0 eliminating those which do not satisfy Condition 2. For the resulting brave-justification structure, we check whether it contains a tuple $(p, L_{\text{now}}, F_{\text{now}}, F_{\text{prev}}, F_{\text{next}})$ such that $\phi \in F_{\text{now}}$. If yes, we return "*entailed at time point p*", otherwise, we return "*not entailed at time point p*". Since the size of J is linear in n, this process requires at most quadratically many steps. The verification that a given tuple is justified requires polynomial time w.r.t. data complexity (the verification of Condition 3 or Condition 2 is linear in n, and only the brave entailment of a BCQ from a DL-Lite$_{\mathcal{R}}$ KB for Condition 1 depends on the size of the ABox, which can be performed in AC^0 w.r.t. data complexity). Therefore, the complete procedure runs in polynomial time w.r.t. data complexity. □

The following theorem summarizes the complexity results for the case without negation in the TCQ.

Theorem 8.4. *The results in Figure 5 hold.*

9. Toward practical algorithms

Until now, work on TCQ answering has primarily focussed on complexity analysis for different DL languages [5, 6, 21]. Attempts towards practical algorithms or implementations are as of now scarce [42, 45]. The only attempt toward more practical algorithms close to our scenario that we are aware of

	Data complexity				Combined complexity			
	classical	AR	IAR	brave	classical	AR	IAR	brave
$\mathcal{EL}_\bot$								
$\mathsf{N_{RC}} = \mathsf{N_{RR}} = \emptyset$	P	coNP	coNP	NP	**NP**	$\mathbf{\Pi_2^p}$	$\mathbf{\Delta_2^p[O(\log n)]}$	**NP**
$\mathsf{N_{RC}} \neq \emptyset, \mathsf{N_{RR}} = \emptyset$	**P**	coNP	coNP	**NP**	**NP**	$\mathbf{\Pi_2^p}$	$\mathbf{\Delta_2^p[O(\log n)]}$	**NP**
$\mathsf{N_{RC}} \neq \emptyset, \mathsf{N_{RR}} \neq \emptyset$	**P**	coNP	coNP	**NP**	**NP**	$\mathbf{\Pi_2^p}$	$\mathbf{\Delta_2^p[O(\log n)]}$	**NP**
DL-Lite$_\mathcal{R}$								
$\mathsf{N_{RC}} = \mathsf{N_{RR}} = \emptyset$	in ALOGTIME	coNP	in P	**in P**	**NP**	$\mathbf{\Pi_2^p}$	**NP**	**NP**
$\mathsf{N_{RC}} \neq \emptyset, \mathsf{N_{RR}} = \emptyset$	in ALOGTIME	coNP	in P	NP	**NP**	$\mathbf{\Pi_2^p}$	**NP**	**NP**
$\mathsf{N_{RC}} \neq \emptyset, \mathsf{N_{RR}} \neq \emptyset$	in ALOGTIME	coNP	in P	NP	**NP**	$\mathbf{\Pi_2^p}$	**NP**	**NP**

Figure 5. Data [left] and combined [right] complexity of BTCQ entailment for BTCQs without negation. All results are tight but those preceded by "in" which are upper bounds. The complexities lower than in the case of BTCQs with negation are in bold.

has been made for DL-Lite$_\mathcal{R}$ and TCQs without negation in [42], and partially implemented [46]. Some of the results have then been generalized in [17] to query languages that are rewritable in the atemporal case. In this section, we will mainly focus on DL-Lite$_\mathcal{R}$ TKBs and TCQs without negation, building on this previous work. However, some of our results also apply to other DLs and we will discuss the case of $\mathcal{EL}_\bot$.

Three different algorithms for answering TCQs without negation over DL-Lite$_\mathcal{R}$ TKBs without rigid predicates are provided in [17, 42]. The first approach is to rewrite the TCQ into a query in ATSQL [47], an SQL variant for temporal databases. The second method first rewrites the TCQ into an equivalent TCQ that does not contain future operators, and then iteratively computes the answers for each time point. The third algorithm computes the answers of the TCQ iteratively as well, but does not eliminate the future operators beforehand. For this, it uses a data structure called *answer formulas*, which represents the TCQs in which some parts have already been evaluated. This structure contains sets of already computed answers to subqueries, as well as variables that serve as placeholders for subqueries that have to be evaluated at the next time point.

Our first contribution is a method for handling rigid predicates (both concepts and roles) in polynomial time for TCQ answering over DL-Lite$_\mathcal{R}$ TKBs under the classical semantics. Indeed, [17, 42] consider only rigid concepts (but not rigid roles) for which they provide a method that is restricted to TCQs that are rooted, i.e., in which each CQ contains an individual or an answer variable that is connected to all the other terms through roles. As a second contribution, we show that in the absence of rigid predicates, it is sometimes possible to combine the algorithms for inconsistency-tolerant query answering in the atemporal case with algorithms for temporal query answering in the consistent case in order to perform inconsistency-tolerant temporal query answering.

9.1. TCQ answering under classical semantics in the presence of rigid predicates for DL-Lite$_\mathcal{R}$ and TCQ without negation nor unbounded future operators

In this section, we show how TCQ answering with rigid predicates can be reduced to TCQ answering without rigid predicates, enabling us to use the algorithms that have been proposed for this latter case. In all the section $\mathcal{K}$ is a DL-Lite$_\mathcal{R}$ TKB and ϕ a TCQ without negation nor unbounded future operators ($\bigcirc$, $\Box$, $\Diamond$, U). This restriction amounts to using the setting of [17, 42] in which the semantics is defined w.r.t. finite sequences of interpretations, and is necessary to reduce TCQ answering with rigid predicates to TCQ answering without rigid predicates. Indeed, consider for instance the query $\Box A(a)$. Such a query can be entailed with rigid predicates, e.g., if A is rigid, but not without rigid predicates since for $p > n$, the interpretation of every predicate is empty in the p^{th} component of the canonical model of a TKB without rigid predicates.

To the best our knowledge, the only algorithm that has been proposed for TCQ answering with rigid predicates and aims at practicality is described in [17, 42], and deals only with rigid concepts and rooted TCQs. We briefly describe this algorithm, which aims at handling streaming data by computing the answers to the query at the last available time point. The key idea is to check all sets of potentially rigid concept assertions and test their compatibility with each of the ABoxes from the sequence together with the TBox. Unfortunately, the original algorithms omits the test whether the checked set of rigid concept assertions covers also the rigid information from the tested ABox together with the TBox. As we found this small flaw in the original algorithm, we present here a mended variant.

The algorithm first constructs every possible set $\mathcal{R}$ of assertions built from the rigid concepts and individuals in the TKB $\mathcal{K}$. Note that there are $2^{|\mathsf{N}_{\mathsf{RC}}^{\mathcal{K}}|*|\mathsf{N}_{\mathsf{I}}^{\mathcal{K}}|}$ such sets. It then runs, in parallel, for each such set $\mathcal{R}$ an instance of the algorithm for TCQ answering without rigid predicates on the TKB that is obtained by adding the assertions in $\mathcal{R}$ to every ABox of the TKB. For each time point i, it takes into account the new dataset available by eliminating the incompatible instances, i.e., those for which

1. $\langle \mathcal{T}, \mathcal{A}_i \cup \mathcal{R} \rangle$ is inconsistent, or
2. a rigid concept assertion entailed by $\langle \mathcal{T}, \mathcal{A}_i \rangle$ does not belong to $\mathcal{R}$.[1]

The answers at time point i are then obtained by taking the intersection of the answers returned by all active instances.

We follow a similar idea in the sense that we also add assertions to the TKB that propagate the effects of the rigid predicates. We show that this way, for DL-Lite$_\mathcal{R}$, TCQ answering with rigid predicates can be reduced to TCQ answering without rigid predicates in *polynomial time*.

In order to show that TCQ answering with rigid predicates can be reduced to TCQ answering without, we construct in polynomial time a set of assertions $\mathcal{R}$ that captures all information about rigid concepts and roles that is relevant for consistency checking and TCQ answering. Then, TCQ answering over $\mathcal{K}$ with $\mathsf{N}_{\mathsf{RC}} \neq \emptyset$, $\mathsf{N}_{\mathsf{RR}} \neq \emptyset$ can be performed by TCQ answering over $\langle \mathcal{T}, (\mathcal{A}_i \cup \mathcal{R})_{0 \leqslant i \leqslant n} \rangle$ with $\mathsf{N}_{\mathsf{RC}} = \mathsf{N}_{\mathsf{RR}} = \emptyset$. Without any restriction on the TBox, $\mathcal{R}$ may be infinite, as illustrated in the following example.

Example 9.1. Consider $\mathcal{K}$ with $\mathcal{T} = \{\exists R^- \sqsubseteq \exists R, R \sqsubseteq S\}$, where S is rigid, $\mathcal{A}_0 = \{R(a,b)\}$, and $\mathcal{A}_i = \emptyset$ for $i \in [\![1, n]\!]$.

Every model of $\mathcal{K}$ that respects rigid predicates satisfies $\phi = \exists x_1 ... x_{k+1}.S(x_1, x_2) \wedge ... \wedge S(x_k, x_{k+1})$ for every $k > 0$ and at every time point. Since with $\mathsf{N}_{\mathsf{RC}} = \mathsf{N}_{\mathsf{RR}} = \emptyset$, $\mathcal{K}$ entails such a query only at time point 0, $\mathcal{R}$ should be such that $\langle \mathcal{T}, \mathcal{R} \rangle$ entails such a query, so that $\langle \mathcal{T}, (\mathcal{A}_i \cup \mathcal{R})_{0 \leqslant i \leqslant n} \rangle$ entails it at every time point. Moreover, there exist models of $\mathcal{K}$ that respect rigid predicates and for which neither $\exists x_1 ... x_k.S(x_1, x_2) \wedge ... \wedge S(x_k, x_1)$ nor $\exists xy.R(x, y)$ hold at any time point $i > 0$. Therefore, $\mathcal{R}$ cannot contain cycles of S, nor R-assertions . Consequently, $\mathcal{R}$ has to contain an infinite chain of S-assertions.

This problem motivates us to disallow rigid roles that have non-rigid sub-roles. In other words, we restrict ourselves in the following to TBoxes $\mathcal{T}$ that entail no role inclusions of the form $P_1 \sqsubseteq P_2$ with $P_1 := R_1 | R_1^-$, $R_1 \in \mathsf{N}_{\mathsf{R}} \backslash \mathsf{N}_{\mathsf{RR}}$ and $P_2 := R_2 | R_2^-$, $R_2 \in \mathsf{N}_{\mathsf{RR}}$. This condition avoids chains of rigid roles in the anonymous part of the canonical model $\mathcal{J}_\mathcal{K}$ that cannot be entailed by a single rigid assertion. In the example above, if rigid roles are only allowed to have rigid sub-roles, then R has to be rigid. In this case, adding the single assertion $R(x, y)$ to every $\mathcal{A}_i$ is sufficient for $\exists x_1 ... x_{k+1}.R(x_1, x_2) \wedge ... \wedge R(x_k, x_{k+1})$ to be entailed at every time point and for every $k > 0$.

As a first step, we explicitly construct the canonical model $\mathcal{J}_\mathcal{K}$ of the DL-Lite$_\mathcal{R}$ TKB $\mathcal{K}$. This model will be used to prove that $\mathcal{K}$ with $\mathsf{N}_{\mathsf{RC}} \neq \emptyset$, $\mathsf{N}_{\mathsf{RR}} \neq \emptyset$ and $\langle \mathcal{T}, (\mathcal{A}_i \cup \mathcal{R})_{0 \leqslant i \leqslant n} \rangle$ with $\mathsf{N}_{\mathsf{RC}} = \mathsf{N}_{\mathsf{RR}} = \emptyset$ entail the same BTCQs without negation nor unbounded future operators.

We build a sequence of (possibly infinite) ABoxes $(chase^{\mathcal{K}}_{\mathrm{rig}}(\mathcal{A}_i))_{0 \leqslant i \leqslant n+1}$ similar to the chase presented in [48] for KBs. Let $\mathcal{S}$ be a set of DL-Lite$_\mathcal{R}$ assertions. We say a PI α is *applicable in* $\mathcal{S}$ *to an assertion* $\beta \in \mathcal{S}$ if one of the following conditions is satisfied:

- $\alpha = A_1 \sqsubseteq A_2, \beta = A_1(a)$ and $A_2(a) \notin \mathcal{S}$,
- $\alpha = A \sqsubseteq \exists P, \beta = A(a)$ and there exists no b such that $P(a, b) \in \mathcal{S}$,
- $\alpha = \exists P \sqsubseteq A, \beta = P(a, b)$ and $A(a) \notin \mathcal{S}$
- $\alpha = \exists P_1 \sqsubseteq \exists P_2, \beta = P_1(a_1, a_2)$ and there exists no b such that $P_2(a_1, b) \in \mathcal{S}$, or
- $\alpha = P_1 \sqsubseteq P_2, \beta = P_1(a_1, a_2)$, and $P_2(a_1, a_2) \notin \mathcal{S}$.

A PI α is *applied to an assertion* β by adding a new assertion β_{new} to $\mathcal{S}$ such that α is not applicable to β in $\mathcal{S} \cup \{\beta_{\text{new}}\}$ anymore.

Definition 9.2 (Rigid chase of a TKB)**.** Let $\mathcal{K} = \langle \mathcal{T}, (\mathcal{A}_i)_{0 \leqslant i \leqslant n} \rangle$ be a DL-Lite$_\mathcal{R}$ TKB. Let $(\mathcal{A}'_i)_{0 \leqslant i \leqslant n+1}$

[1] This condition is new and added after consultation with the authors of [17, 42].

be such that $\mathcal{A}'_i = \mathcal{A}_i \cup \{\beta \mid \exists k, \beta \in \mathcal{A}_k \text{ and } \beta \text{ is rigid}\}$ for $i \in [\![0,n]\!]$ and $\mathcal{A}'_{n+1} = \emptyset$. Finally, let $\mathcal{T}_p$ be the set of positive inclusions in $\mathcal{T}$, and N_i be the number of assertions in $\mathcal{A}'_i$. Assume that the assertions in each $\mathcal{A}'_i$ are enumerated from $N_1+\cdots+N_{i-1}+1$ to $N_1+\cdots+N_i$ following their lexicographic order. Consider the sequences of sets $\mathcal{S}^j = (\mathcal{S}^j_i)_{0\leqslant i\leqslant n+1}$ of assertions defined by

$$\mathcal{S}^0 = (\mathcal{A}'_i)_{0\leqslant i\leqslant n+1}$$
$$\mathcal{S}^{j+1} = \mathcal{S}^j \cup \mathcal{S}^{\text{new}} = (\mathcal{S}^j_i \cup \mathcal{S}^{\text{new}}_i)_{0\leqslant i\leqslant n+1},$$

where $\mathcal{S}^{\text{new}}$ is defined in terms of the assertion β_{new} obtained as follows: let $\beta \in \mathcal{S}^j_{i_\beta}$ be the first assertion in $\mathcal{S}^j$ such that there exists a PI in $\mathcal{T}_p$ applicable in $\mathcal{S}^j_{i_\beta}$ to β and let α be the lexicographically first PI applicable in $\mathcal{S}^j_{i_\beta}$ to β. In case α, β are of the form

- $\alpha = A_1 \sqsubseteq A_2$ and $\beta = A_1(a)$ then $\beta_{\text{new}} = A_2(a)$
- $\alpha = A \sqsubseteq \exists P$ and $\beta = A(a)$ then $\beta_{\text{new}} = P(a, a_{\text{new}})$
- $\alpha = \exists P \sqsubseteq A$ and $\beta = P(a,b)$ then $\beta_{\text{new}} = A(a)$
- $\alpha = \exists P_1 \sqsubseteq \exists P$ and $\beta = P_1(a,b)$ then $\beta_{\text{new}} = P(a, a_{\text{new}})$
- $\alpha = P_1 \sqsubseteq P_2$ and $\beta = P_1(a_1,a_2)$ then $\beta_{\text{new}} = P_2(a_1,a_2)$

where a_{new} is constructed from α and β as follows:

- if $a \in \mathsf{N}^{\mathcal{K}}_{\mathsf{I}}$ then $a_{\text{new}} = x^{i_\beta}_{aP}$
- otherwise $a \notin \mathsf{N}^{\mathcal{K}}_{\mathsf{I}}$, then let $a = x^{i_1\ldots i_l}_{a'P_1\ldots P_l}$ and define $a_{\text{new}} = x^{i_1\ldots i_l i_\beta}_{a'P_1\ldots P_l P}$.

If β_{new} is rigid, then $\mathcal{S}^{\text{new}} = (\{\beta_{\text{new}}\})_{0\leqslant i\leqslant n+1}$, otherwise, $\mathcal{S}^{\text{new}} = (\mathcal{S}^{\text{new}}_i)_{0\leqslant i\leqslant n+1}$ with $\mathcal{S}^{\text{new}}_{i_\beta} = \{\beta_{\text{new}}\}$ and $\mathcal{S}^{\text{new}}_i = \emptyset$ for $i \neq i_\beta$.

Let N be the total number of assertions in $\mathcal{S}^j$. The assertion(s) added are numbered as follows: if β_{new} is not rigid, β_{new} is numbered by $N+1$, otherwise for every $i \in [\![0,n+1]\!]$, the assertion $\beta_{\text{new}} \in \mathcal{S}^{\text{new}}_i$ added to $\mathcal{S}^j_i$ is numbered by $N+1+i$.

We call the rigid chase of $\mathcal{K}$, denoted by $\mathit{chase}_{\text{rig}}(\mathcal{K}) = (\mathit{chase}^{\mathcal{K}}_{\text{rig}}(\mathcal{A}_i))_{0\leqslant i\leqslant n+1}$, the sequence of sets of assertions obtained as the infinite union of all $\mathcal{S}^j$, i.e.,

$$(\mathit{chase}^{\mathcal{K}}_{\text{rig}}(\mathcal{A}_i))_{0\leqslant i\leqslant n+1} = (\bigcup_{j\in\mathbb{N}} \mathcal{S}^j_i)_{0\leqslant i\leqslant n+1}.$$

Based on the rigid chase of $\mathcal{K}$, we construct the sequence of interpretations $\mathcal{J}_{\mathcal{K}} = (\mathcal{I}_i)_{i\geqslant 0}$, where $\mathcal{I}_i = \langle \Delta, \cdot^{\mathcal{I}_i}\rangle$ is defined as follows.

- $\Delta = \mathsf{N}^{\mathcal{K}}_{\mathsf{I}} \cup \Gamma_N$, where Γ_N is the set of individuals that appear in $\mathit{chase}_{\text{rig}}(\mathcal{K})$ and not in $\mathcal{K}$.
- For every $a \in \Delta$, $a^{\mathcal{I}_i} = a$.
- For every $A \in \mathsf{N}_{\mathsf{C}}$, $A^{\mathcal{I}_i} = \{a \mid A(a) \in \mathit{chase}^{\mathcal{K}}_{\text{rig}}(\mathcal{A}_i)\}$ if $i \leqslant n$, $A^{\mathcal{I}_i} = \{a \mid A(a) \in \mathit{chase}^{\mathcal{K}}_{\text{rig}}(\mathcal{A}_{n+1})\}$ if $i > n$.
- For every $R \in \mathsf{N}_{\mathsf{R}}$, $R^{\mathcal{I}_i} = \{(a,b) \mid R(a,b) \in \mathit{chase}^{\mathcal{K}}_{\text{rig}}(\mathcal{A}_i)\}$ if $i \leqslant n$, $R^{\mathcal{I}_i} = \{(a,b) \mid R(a,b) \in \mathit{chase}^{\mathcal{K}}_{\text{rig}}(\mathcal{A}_{n+1})\}$ if $i > n$.

We show that $\mathcal{J}_{\mathcal{K}}$ is a model of $\mathcal{K}$ that respects rigid predicates, and that for any BTCQ without negation ϕ such that $\mathsf{N}^{\phi}_{\mathsf{I}} \subseteq \mathsf{N}^{\mathcal{K}}_{\mathsf{I}}$, $\mathcal{K}, p \models \phi$ iff $\mathcal{J}_{\mathcal{K}}, p \models \phi$.

Lemma 9.3. *If $\mathcal{K}$ is consistent, then $\mathcal{J}_{\mathcal{K}}$ is a model of $\mathcal{K}$ that respects rigid predicates.*

Proof (Sketch). Since for every $i \in [\![0,n]\!]$, $\mathcal{A}_i \subseteq \mathit{chase}^{\mathcal{K}}_{\text{rig}}(\mathcal{A}_i)$, we directly obtain $\mathcal{I}_i \models \mathcal{A}_i$. We can show that for every i, $\mathcal{I}_i$ further satisfies every positive inclusion of $\mathcal{T}$ with similar arguments as those used in [48]. Indeed, every PI applicable to an assertion β in $\mathcal{S}^j_i$ at step j of the construction of the rigid chase becomes not applicable to β in $\mathcal{S}^k_i$ for some $k \geqslant j$, because there are neither infinitely many assertions before β, nor infinitely many PIs applied to some assertion that precedes β. Finally, we show that $\mathcal{I}_i$ satisfies every negative inclusion of $\mathcal{T}$ because otherwise $\mathcal{K}$ would be inconsistent. Moreover, the model $\mathcal{J}_{\mathcal{K}}$ respects rigid predicates because, if an assertion β of $\mathit{chase}^{\mathcal{K}}_{\text{rig}}(\mathcal{A}_i)$ is rigid, either $\beta \in \mathcal{A}_i$ and by construction $\beta \in \mathcal{S}^0_k = \mathcal{A}'_k$ for every k, or β has been derived at some step j by applying some PI to an assertion of $\mathcal{S}^j$ and $\beta \in \mathcal{S}^{j+1}_k$ for every k. We obtain that in both cases, $\beta \in \mathit{chase}^{\mathcal{K}}_{\text{rig}}(\mathcal{A}_k)$ for every k. □

Next, we show that $\mathcal{J}_{\mathcal{K}}$ is the canonical model of $\mathcal{K}$ for entailment of BTCQ without negation.

Lemma 9.4. *If $\mathcal{K}$ is consistent, then for every BTCQ without negation ϕ such that $\mathsf{N}^{\phi}_{\mathsf{I}} \subseteq \mathsf{N}^{\mathcal{K}}_{\mathsf{I}}$, $\mathcal{K}, p \models \phi$ iff $\mathcal{J}_{\mathcal{K}}, p \models \phi$.*

Proof (Sketch). Since $\mathcal{J}_{\mathcal{K}} = (\mathcal{I}_i)_{i\geqslant 0}$ with $\mathcal{I}_i = \langle \Delta, \cdot^{\mathcal{I}_i}\rangle$ is a model of $\mathcal{K}$ that respects rigid predicates, the first direction is straightforward, and we only need to show that $\mathcal{J}_{\mathcal{K}}, p \models \phi$ implies $\mathcal{K}, p \models \phi$. Let $\mathcal{J} = (\mathcal{I}'_i)_{i\geqslant 0}$ with $\mathcal{I}'_i = \langle \Delta', \cdot^{\mathcal{I}'_i}\rangle$ be a model of $\mathcal{K}$ that respects rigid predicates. We show by structural induction on ϕ that if $\mathcal{J}_{\mathcal{K}}, p \models \phi$, then $\mathcal{J}, p \models \phi$. For the case where ϕ is a CQ $\exists\vec{y}.\psi(\vec{y})$, we show that if there exists a homomorphism π of $\exists\vec{y}.\psi(\vec{y})$ into $\mathcal{I}_p$, then $\mathcal{I}'_p \models \exists\vec{y}.\psi(\vec{y})$, by defining a homomorphism h from Δ into Δ'. □

We are now ready to introduce the set $\mathcal{R}$ that, if added to every ABox of the TKB, allows us to reduce TCQ answering with rigid predicates to TCQ answering without.

Proposition 9.5. *Let $\mathcal{R}$ be as defined in Figure 6. The set $\mathcal{R}$ is computable in polynomial time and such that*

1. *$\mathcal{K}$ is consistent iff $\mathcal{K}_\mathcal{R} = \langle \mathcal{T}, (\mathcal{A}_i \cup \mathcal{R})_{0\leqslant i\leqslant n}\rangle$ is consistent with $\mathsf{N}_{\mathsf{RC}} = \mathsf{N}_{\mathsf{RR}} = \emptyset$, and*
2. *for any BTCQ ϕ without negation nor unbounded future operators and such that $\mathsf{N}_\mathsf{I}^\phi \subseteq \mathsf{N}_\mathsf{I}^\mathcal{K}$, $\mathcal{K}, p \models \phi$ iff $\mathcal{K}_\mathcal{R}, p \models \phi$ with $\mathsf{N}_{\mathsf{RC}} = \mathsf{N}_{\mathsf{RR}} = \emptyset$.*

The size of $\mathcal{R}$ is polynomial in the size of $\mathsf{N}_\mathsf{C}^\mathcal{K}, \mathsf{N}_\mathsf{R}^\mathcal{K}$, and $\mathsf{N}_\mathsf{I}^\mathcal{K}$, and since query answering as well as subsumption checking are in P, $\mathcal{R}$ can be computed in polynomial time. The first three parts of $\mathcal{R}$ contain information about the participation of individuals of $\mathsf{N}_\mathsf{I}^\mathcal{K}$ in rigid predicates. The last two witness the participation in rigid predicates of the role-successors w.r.t. non-rigid roles, thus take into account also anonymous individuals that are created in $chase_{\mathrm{rig}}(\mathcal{K})$ when applying PIs whose right-hand side is an existential restriction of a non-rigid role. Note that the individuals created in $chase_{\mathrm{rig}}(\mathcal{K})$ when applying such a PI with a rigid role are witnessed by the individuals x_{aP} or $x_{P_1P_2}$ if they do not follow from a rigid role assertion, and do not need to be witnessed otherwise, since the assertion $P_2(x_{P_1}, x_{P_1P_2})$ is sufficient to trigger the generation of the whole anonymous part implied by the fact that $x_{P_1P_2}$ is in the range of P_2.

We break the proof of Proposition 9.5 into several lemmas.

Lemma 9.6. *$\mathcal{K}$ is consistent iff $\mathcal{K}_\mathcal{R}$ is consistent with $\mathsf{N}_{\mathsf{RC}} = \mathsf{N}_{\mathsf{RR}} = \emptyset$.*

Proof. By Proposition 3.5, $\mathcal{K}_\mathcal{R}$ is consistent with $\mathsf{N}_{\mathsf{RC}} = \mathsf{N}_{\mathsf{RR}} = \emptyset$ iff each $\langle \mathcal{T}, \mathcal{A}_i \cup \mathcal{R}\rangle$ is consistent. We show that also $\mathcal{K}$ is consistent iff each $\langle \mathcal{T}, \mathcal{A}_i \cup \mathcal{R}\rangle$ is consistent.

If $\mathcal{K}$ is not consistent, let $\mathcal{B}$ be a minimal inconsistent subset of $\mathcal{K}$. Then $\mathcal{B}$ is either internal to some $\mathcal{A}_i$, and $\langle \mathcal{T}, \mathcal{A}_i \cup \mathcal{R}\rangle$ is inconsistent, or is of the form $\mathcal{B} = \{(\alpha, i), (\beta, j)\}$ with $i \neq j$. In the latter case, $\{\alpha,\beta\}$ violates some negative inclusion in the closure of the TBox that involves at least a rigid concept A or a rigid role R by assigning an individual a (or two individuals a, b) to two disjoint concepts (or roles). We can then assume w.l.o.g. that we are in one of the following cases: (i) $\langle \mathcal{T}, \alpha\rangle \models A(a)$, (ii) $\langle \mathcal{T}, \alpha\rangle \models \exists x.R(a, x)$, (iii) $\langle \mathcal{T}, \alpha\rangle \models \exists x.R(x, a)$, or (iv) $\langle \mathcal{T}, \alpha\rangle \models R(a, b)$. It follows that respectively (i) $\langle \mathcal{T}, \mathcal{A}_i\rangle \models A(a)$, (ii) $\langle \mathcal{T}, \mathcal{A}_i\rangle \models \exists x.R(a, x)$, (iii) $\langle \mathcal{T}, \mathcal{A}_i\rangle \models \exists x.R(x, a)$, or (iv) $\langle \mathcal{T}, \mathcal{A}_i\rangle \models R(a, b)$. By construction of $\mathcal{R}$, we then conclude that (i) $A(a) \in \mathcal{R}$, (ii) $R(a, x_{aR}) \in \mathcal{R}$, (iii) $R(x_{aR^-}, a) \in \mathcal{R}$, or (iv) $R(a, b) \in \mathcal{R}$ respectively, and therefore $\langle \mathcal{T}, \mathcal{A}_j \cup \mathcal{R}\rangle$ is inconsistent.

In the other direction, assume there exists $i \in [\![0, n]\!]$, such that $\langle \mathcal{T}, \mathcal{A}_i \cup \mathcal{R}\rangle$ is inconsistent, and let $\mathcal{B}$ be a minimal inconsistent subset of $\langle \mathcal{T}, \mathcal{A}_i \cup \mathcal{R}\rangle$. If $\mathcal{B}$ is internal to $\mathcal{A}_i$, $\mathcal{K}$ is clearly inconsistent. Otherwise, $\mathcal{B}$ is of the form $\{\alpha,\beta\}$ and involves at least one assertion from $\mathcal{R}$. The assertions α and β assign an individual x to two disjoint concepts C_1, C_2, or two individuals x, y to two disjoint roles R_1, R_2. We distinguish three cases. In the case where $x = x_{aP}$ (resp. $x = x_{P_1P_2}$), since $P(a, x_{aP})$ (resp. $P_2(x_{P_1}, x_{P_1P_2})$) is the only assertion of $\mathcal{R}$ that contains x, we obtain that $\exists P^-$ (resp. $\exists P_2^-$) is unsatisfiable. Since there exists j such that $\langle \mathcal{T}, \mathcal{A}_j\rangle \models \exists x.P(a, x)$ (resp. $\langle \mathcal{T}, \mathcal{A}_j\rangle \models \exists xy.P_1(x, y)$ and $\mathcal{T} \models \exists P_1^- \sqsubseteq \exists P_2$), it follows that $\mathcal{A}_j$ is inconsistent. In the case where $x = x_{P_1}$, since x_{P_1} appears only in concepts that subsume $\exists P_1^-$, the fact that x is assigned to two disjoint concepts implies that $\exists P_1^-$ is unsatisfiable. Therefore, and since there exists j such that $\langle \mathcal{T}, \mathcal{A}_j\rangle \models \exists xy.P_1(x, y)$, $\mathcal{A}_j$ is inconsistent. Finally, in the case where $x \in \mathsf{N}_\mathsf{I}^\mathcal{K}$, since α or β is in $\mathcal{R}$, at least one of C_1, C_2 (or R_1, R_2) is rigid. If some $\mathcal{A}_j$ is inconsistent, so is $\mathcal{K}$. Otherwise, every $\mathcal{A}_j$ is consistent. If $\alpha \notin \mathcal{A}_i$, let $c_\alpha \in \mathcal{A}_{j_\alpha}$ be an assertion responsible for the entailment that triggered the addition of α to $\mathcal{R}$, and otherwise let $(c_\alpha, j_\alpha) = (\alpha, i)$. If $\beta \notin \mathcal{A}_i$, let $c_\beta \in \mathcal{A}_{j_\beta}$ be an assertion responsible for the entailment that triggered the addition of β to $\mathcal{R}$, and otherwise $(c_\beta, j_\beta) = (\beta, i)$. Then $\{(c_\alpha, j_\alpha), (c_\beta, j_\beta)\}$ is inconsistent because c_α and c_β lead to a (or a, b) being assigned to two disjoint concepts (or disjoint roles) such that at least one of them is rigid. □

We now assume that $\mathcal{K}$ and $\mathcal{K}_\mathcal{R}$ are consistent. Note that if this is not the case, they both trivially entail any BTCQ. The two following lemmas show that if a Boolean conjunctive query $q = \exists \vec{y}.\psi(\vec{y})$ is such that $\mathsf{N}_\mathsf{I}^q \subseteq \mathsf{N}_\mathsf{I}^\mathcal{K}$, then for every $p \in [\![0, n]\!]$, $\mathcal{K}_\mathcal{R}, p \models q$ iff $\mathcal{K}, p \models q$.

Lemma 9.7. *Let $q = \exists \vec{y}.\psi(\vec{y})$ be such that $\mathsf{N}_\mathsf{I}^q \subseteq \mathsf{N}_\mathsf{I}^\mathcal{K}$. For every $p \in [\![0, n]\!]$, if $\mathcal{K}_\mathcal{R}, p \models q$ then $\mathcal{K}, p \models q$.*

Proof (Sketch). This lemma can be shown by defining a homomorphism from the canonical model of $\langle \mathcal{T}, \mathcal{A}_p \cup \mathcal{R}\rangle$ into $\mathcal{I}_p$. □

$$\begin{aligned}\mathcal{R} = &\{A(a) \mid A \in \mathsf{N}^{\mathcal{K}}_{\mathsf{RC}}, a \in \mathsf{N}^{\mathcal{K}}_{\mathsf{I}}, \exists i, \langle \mathcal{T}, \mathcal{A}_i \rangle \models A(a)\} \cup \\ &\{R(a,b) \mid R \in \mathsf{N}^{\mathcal{K}}_{\mathsf{RR}}, a, b \in \mathsf{N}^{\mathcal{K}}_{\mathsf{I}}, \exists i, \langle \mathcal{T}, \mathcal{A}_i \rangle \models R(a,b)\} \cup \\ &\{P(a, x_{aP}) \mid R \in \mathsf{N}^{\mathcal{K}}_{\mathsf{RR}}, P := R|R^-, a \in \mathsf{N}^{\mathcal{K}}_{\mathsf{I}}, \exists i, \langle \mathcal{T}, \mathcal{A}_i \rangle \models \exists x.P(a,x)\} \cup \\ &\{A(x_{P_1}) \mid S \in \mathsf{N}^{\mathcal{K}}_{\mathsf{R}} \backslash \mathsf{N}^{\mathcal{K}}_{\mathsf{RR}}, P_1 := S|S^-, A \in \mathsf{N}^{\mathcal{K}}_{\mathsf{RC}}, \exists i, \langle \mathcal{T}, \mathcal{A}_i \rangle \models \exists xy.P_1(x,y) \text{ and } \mathcal{T} \models \exists P_1^- \sqsubseteq A\} \cup \\ &\{P_2(x_{P_1}, x_{P_1P_2}) \mid S \in \mathsf{N}^{\mathcal{K}}_{\mathsf{R}} \backslash \mathsf{N}^{\mathcal{K}}_{\mathsf{RR}}, P_1 := S|S^-, R \in \mathsf{N}^{\mathcal{K}}_{\mathsf{RR}}, P_2 := R|R^-, \exists i, \langle \mathcal{T}, \mathcal{A}_i \rangle \models \exists xy.P_1(x,y) \\ &\quad \text{and } \mathcal{T} \models \exists P_1^- \sqsubseteq \exists P_2\}\end{aligned}$$

Figure 6. Set of rigid assertions added to every ABox of $\mathcal{K}$

Lemma 9.8. *Let $q = \exists \vec{y}.\psi(\vec{y})$ be such that $\mathsf{N}^q_{\mathsf{I}} \subseteq \mathsf{N}^{\mathcal{K}}_{\mathsf{I}}$. For every $p \in [\![0,n]\!]$, if $\mathcal{K}, p \models q$ then $\mathcal{K}_{\mathcal{R}}, p \models q$.*

Proof (Sketch). The lemma can be shown by considering a model $\mathcal{I}^{\mathcal{R}}_p$ of $\langle \mathcal{T}, \mathcal{A}_p \cup \mathcal{R} \rangle$, and defining a homomorphism of $\mathcal{I}_p$ into $\mathcal{I}^{\mathcal{R}}_p$. □

Since by Lemmas 9.7 and 9.8, $\mathcal{K}$ and $\mathcal{K}_{\mathcal{R}}$ with $\mathsf{N}_{\mathsf{RC}} = \mathsf{N}_{\mathsf{RR}} = \emptyset$ coincide on the entailment of BCQs for every time point $p \in [\![0,n]\!]$, we can show as in Proposition 7.6 that they coincide on entailment of BTCQs without negation nor unbounded future operators.

Lemma 9.9. *Let ϕ be a BTCQ without negation nor unbounded future operators and such that $\mathsf{N}^{\phi}_{\mathsf{I}} \subseteq \mathsf{N}^{\mathcal{K}}_{\mathsf{I}}$. $\mathcal{K}, p \models \phi$ iff $\mathcal{K}_{\mathcal{R}}, p \models \phi$ with $\mathsf{N}_{\mathsf{RC}} = \mathsf{N}_{\mathsf{RR}} = \emptyset$.*

It follows that TCQs can be answered in $\mathcal{K}$ with rigid predicates by answering TCQs in $\mathcal{K}_{\mathcal{R}}$ without rigid predicates and pruning answers that contain individual names not in $\mathsf{N}^{\mathcal{K}}_{\mathsf{I}}$. Note that every model of $\mathcal{K}_{\mathcal{R}}$ is a model of $\mathcal{K}$, but does not respect rigid predicates in general. We can reduce BTCQ entailment over $\mathcal{K}$ with rigid predicates to BTCQ entailment over $\mathcal{K}_{\mathcal{R}}$ without rigid predicates only because our TCQs do not allow LTL operators to be nested in existential quantifications. This prevents existentially quantified variables to link different time points. To see this, consider the query $\exists xy.\Box^b(R(a,x) \wedge R(x,y))$ and the TKB $\mathcal{K} = \langle \mathcal{T}, (\mathcal{A}_i)_{0 \leqslant i \leqslant n} \rangle$ with $\mathcal{T} = \{B \sqsubseteq \exists R, \exists R^- \sqsubseteq \exists R\}$, $R \in \mathsf{N}_{\mathsf{RR}}$ and $\mathcal{A}_i = \{B(a)\}$. For this TKB, we would have $\mathcal{R} = \{R(a, x_{aR})\}$, and therefore x_{aR} could have a different R-successors in each interpretation of a model of $\mathcal{K}_{\mathcal{R}}$, thus y cannot be mapped to the same object at every time point.

Remark 9.10. In the case of streaming data, if we want to take into account a newly available dataset, we do not need to fully recompute $\mathcal{R}$: we only need to add the new rigid assertions that can be derived from the new dataset. Moreover, if we only reason over a window of n time points from our stream, we can annotate the assertions in $\mathcal{R}$ with a counter that is initialised with n and decremented with each new time point. Assertions are then removed from $\mathcal{R}$ if their counter reaches 0. Here, we implicitly assume that the counter for an assertion is reset to n whenever it is again derived from the next dataset.

Remark 9.11. The main goal of the approaches presented in [17, 42] for TCQ answering in DL-Lite$_{\mathcal{R}}$ is to obtain the query answers at the last time point without storing all the data for all previous time points. Their algorithm uses a *bounded history encoding*, which means that the space required by the algorithm is constant w.r.t. the number n of previous time points: only the current dataset and some auxiliary relations required for computing the query answers are stored and updated at each time point.

Unfortunately, with rigid predicates present, our approach does not achieve bounded history encoding, since the answers of the subqueries of ϕ at previous time points may change when new rigid assertions are derived from the last dataset. However, if the algorithm of [17, 42] has this property, it requires exponential space w.r.t. $\mathsf{N}^{\mathcal{K}}_{\mathsf{RC}}$ and $\mathsf{N}^{\mathcal{K}}_{\mathsf{I}}$ which can also be problematic, while our algorithm requires only polynomial space and time. To achieve bounded history encoding (but in exponential time w.r.t. $\mathsf{N}^{\mathcal{K}}_{\mathsf{RC}}$, $\mathsf{N}^{\mathcal{K}}_{\mathsf{RR}}$ and $\mathsf{N}^{\mathcal{K}}_{\mathsf{I}}$), we could adapt the algorithm of [17, 42] to support rigid roles. We would consider all possible sets $\mathcal{R}$ built from $\mathsf{N}^{\mathcal{K}}_{\mathsf{RC}}$, $\mathsf{N}^{\mathcal{K}}_{\mathsf{RR}}$ and $\mathsf{N}^{\mathcal{K}}_{\mathsf{I}}$ following the form of Figure 6, then verify at each time point whether $\mathcal{R}$ is consistent with

$\mathcal{A}_i$ and $\mathcal{T}$ and contains all rigid assertions that can be derived from $\mathcal{A}_i$ as described in Figure 6.

A possible direction to alleviate the restrictions on the TBox that forbid rigid roles that have non-rigid sub-roles would be to use ideas similar to those developed in [43] for CQ answering over DL-Lite$_\mathcal{R}$ KB using the combined approach. This CQ answering approach saturates the data by adding to the ABox every assertion that can be derived, introducing individual names to witness existential role restrictions, and then uses a special rewriting to prune spurious answers. In our setting, we could model infinite chains of rigid roles by adding cycles of rigid roles to $\mathcal{R}$, then prune the spurious answers resulting from these cycles.

Regarding $\mathcal{EL}_\bot$, we conjecture that we could have a similar approach for rigid predicates. The main difference would be that since in $\mathcal{EL}_\bot$ several assertions may be needed to derive one, $\mathcal{R}$ would have to be computed iteratively, taking into account its own assertions to derive new ones until a fix-point is reached. Moreover, the problem of infinite chains of rigid roles that cannot be entailed by a polynomial set of assertions would appear as soon as $\mathsf{N_{RR}} \neq \emptyset$. The combined approach for $\mathcal{EL}$ [11] could provide ideas to overcome this difficulty.

9.2. *Inconsistency-tolerant TCQ answering without rigid predicates*

In this section $\mathcal{K}$ is a $\mathcal{L}$ TKB and ϕ a TCQ without negation.

When $\mathsf{N_{RC}} = \mathsf{N_{RR}} = \emptyset$, an important consequence of Proposition 3.5 is that the repairs of $\mathcal{K}$ are all possible sequences $(\mathcal{A}'_i)_{0\leqslant i\leqslant n}$ where $\mathcal{A}'_i$ is a repair of $\langle\mathcal{T},\mathcal{A}_i\rangle$, so the intersection of the repairs of $\mathcal{K}$ is $(\mathcal{A}^\cap_i)_{0\leqslant i\leqslant n}$ where $\mathcal{A}^\cap_i$ is the intersection of the repairs of $\langle\mathcal{T},\mathcal{A}_i\rangle$. This allow us to show that the entailment (resp. IAR entailment) of a BTCQ without negation from a consistent (resp. possibly inconsistent) TKB in a DL $\mathcal{L}$ that has the canonical model property for BCQ entailment can be equivalently defined w.r.t. the entailment (resp. IAR entailment) of the BCQs it contains as follows:

Proposition 9.12. *If $\mathcal{L}$ has the canonical model property for BCQ entailment and $\mathsf{N_{RC}} = \mathsf{N_{RR}} = \emptyset$, then the entailments shown in Table 3 hold for S = classical when $\mathcal{K}$ is consistent, and for S = IAR.*

Table 3

Entailment under classical or IAR semantics from a $\mathcal{L}$ TKB without rigid predicates and such that $\mathcal{L}$ has the canonical model property for BCQ entailment

ϕ	$\mathcal{K},p\models_S\phi$ iff
$\exists\vec{y}.\psi(\vec{y})$	$p\leqslant n$ and $\langle\mathcal{T},\mathcal{A}_p\rangle\models_S\exists\vec{y}.\psi(\vec{y})$
$\phi_1\wedge\phi_2$	$\mathcal{K},p\models_S\phi_1$ and $\mathcal{K},p\models_S\phi_2$
$\phi_1\vee\phi_2$	$\mathcal{K},p\models_S\phi_1$ or $\mathcal{K},p\models_S\phi_2$
$\bigcirc\phi_1$	$\mathcal{K},p+1\models_S\phi_1$
$\bullet^b\phi_1$	$p<n$ implies $\mathcal{K},p+1\models_S\phi_1$
$\bigcirc^-\phi_1$	$p>0$ and $\mathcal{K},p-1\models_S\phi_1$
$\bullet^-\phi_1$	$p>0$ implies $\mathcal{K},p-1\models_S\phi_1$
$\Box\phi_1$	$\forall k,k\geqslant p,\mathcal{K},k\models\phi_1$
$\Box^b\phi_1$	$\forall k,p\leqslant k\leqslant n,\mathcal{K},k\models\phi_1$
$\Box^-\phi_1$	$\forall k,0\leqslant k\leqslant p,\mathcal{K},k\models_S\phi_1$
$\Diamond\phi_1$	$\exists k,k\geqslant p,\mathcal{K},k\models\phi_1$
$\Diamond^b\phi_1$	$\exists k,p\leqslant k\leqslant n,\mathcal{K},k\models\phi_1$
$\Diamond^-\phi_1$	$\exists k,0\leqslant k\leqslant p,\mathcal{K},k\models_S\phi_1$
$\phi_1\mathsf{U}\phi_2$	$\exists k,k\geqslant p,\mathcal{K},k\models\phi_2$ and $\forall j,p\leqslant j<k,\mathcal{K},j\models\phi_1$
$\phi_1\mathsf{U}^b\phi_2$	$\exists k,p\leqslant k\leqslant n,\mathcal{K},k\models_S\phi_2$ and $\forall j,p\leqslant j<k,\mathcal{K},j\models_S\phi_1$
$\phi_1\mathsf{S}\phi_2$	$\exists k,0\leqslant k\leqslant p,\mathcal{K},k\models_S\phi_2$ and $\forall j,k<j\leqslant p,\mathcal{K},j\models_S\phi_1$

Proof. For the consistent case, all relations in Table 3 but the first one are straightforwardly obtained by applying the definitions of BTCQ satisfaction of Table 1 to the canonical model of $\mathcal{K}$. Moreover, by Proposition 3.6, if $p\leqslant n$, then $\mathcal{K},p\models\exists\vec{y}.\psi(\vec{y})$ iff $\langle\mathcal{T},\mathcal{A}_p\rangle\models\exists\vec{y}.\psi(\vec{y})$. Finally, $\mathcal{K},p\not\models\exists\vec{y}.\psi(\vec{y})$ if $p>n$, because there exists a model of $\mathcal{K}$ whose p^{th} component interprets every predicate as the empty set.

For IAR semantics, let $(\mathcal{A}^{\text{ir}}_i)_{0\leqslant i\leqslant n}$ denote the intersection of the repairs of $\mathcal{K}$ and $\mathcal{A}^\cap_i$ denote the intersection of the repairs of $\langle\mathcal{T},\mathcal{A}_i\rangle$.

- $\mathcal{K},p\models_{\text{IAR}}\exists\vec{y}.\psi(\vec{y})$ iff $\langle\mathcal{T},(\mathcal{A}^{\text{ir}}_i)_{0\leqslant i\leqslant n}\rangle,p\models\exists\vec{y}.\psi(\vec{y})$, i.e., iff $p\leqslant n$ and $\langle\mathcal{T},\mathcal{A}^{\text{ir}}_p\rangle\models\exists\vec{y}.\psi(\vec{y})$ because $(\mathcal{A}^{\text{ir}}_i)_{0\leqslant i\leqslant n}$ is consistent. Since the repairs of $\mathcal{K}$ are the sequences of the repairs of the $\langle\mathcal{T},\mathcal{A}_i\rangle$, $\mathcal{A}^{\text{ir}}_p=\mathcal{A}^\cap_p$, so $\mathcal{K},p\models_{\text{IAR}}\exists\vec{y}.\psi(\vec{y})$ iff $p\leqslant n$ and $\langle\mathcal{T},\mathcal{A}_p\rangle\models_{\text{IAR}}\exists\vec{y}.\psi(\vec{y})$.
- $\mathcal{K},p\models_{\text{IAR}}\phi_1\wedge\phi_2$ iff $\langle\mathcal{T},(\mathcal{A}^{\text{ir}}_i)_{0\leqslant i\leqslant n}\rangle,p\models\phi_1\wedge\phi_2$, i.e., iff $\langle\mathcal{T},(\mathcal{A}^{\text{ir}}_i)_{0\leqslant i\leqslant n}\rangle,p\models\phi_1$ and $\langle\mathcal{T},(\mathcal{A}^{\text{ir}}_i)_{0\leqslant i\leqslant n}\rangle,p\models\phi_2$ because $(\mathcal{A}^{\text{ir}}_i)_{0\leqslant i\leqslant n}$ is consistent. It follows that $\mathcal{K},p\models_{\text{IAR}}\phi_1\wedge\phi_2$ iff $\mathcal{K},p\models_{\text{IAR}}\phi_1$ and $\mathcal{K},p\models_{\text{IAR}}\phi_2$.
- We show all remaining relations in the same way, applying the definition of IAR semantics and using the fact that $(\mathcal{A}^{\text{ir}}_i)_{0\leqslant i\leqslant n}$ is consistent.

□

This is a remarkable result, since it implies that answering temporal CQs under IAR semantics can be done with the algorithms developed for the consistent case (see [17, 42] for algorithms for DL-Lite$_\mathcal{R}$ without unbounded future operators) by replacing classical CQ answering by IAR CQ answering (see [29, 36, 37] for algorithms for DL-Lite$_\mathcal{R}$). The following example shows that this is unfortunately not true for brave or AR semantics.

Example 9.13. Consider the TKB $\mathcal{K} = \langle\mathcal{T}, (\mathcal{A}_i)_{0\leqslant i\leqslant n}\rangle$ and TCQ ϕ.

$$\begin{aligned}\mathcal{T} &= \{T \sqsubseteq \neg F\}\\ \mathcal{A}_i &= \{T(a), F(a)\} \text{ for } 0 \leqslant i \leqslant n\\ \phi &= \Box^-(T(a) \wedge \CIRCLE^- F(a))\end{aligned}$$

Now, $\mathcal{K}, k \models_{\text{brave}} T(a) \wedge \CIRCLE^- F(a)$ for every $0 \leqslant k \leqslant n$, but $\mathcal{K}, n \not\models_{\text{brave}} \phi$. This is because the same repair cannot entail $T(a) \wedge \CIRCLE^- F(a)$ both at time point k and $k+1$, since it would contain both $(T(a), k)$ and $(F(a), k)$ which is not possible. For AR semantics, consider $\phi = T(a) \vee F(a)$ over the TKB $\mathcal{K}$: while ϕ holds under AR semantics at each time point, neither $T(a)$ nor $F(a)$ does.

However, if the operators allowed in the TCQ are restricted to $\wedge, \Circle, \CIRCLE^b, \Circle^-, \CIRCLE^-, \Box, \Box^b$, and $\Box^-$, then AR TCQ answering can be done with the algorithms developed for the consistent case by simply replacing classical CQ answering by AR CQ answering (see [37] for algorithms for DL-Lite$_\mathcal{R}$). Indeed, for these operators, the relations of Proposition 9.12 hold for $\mathsf{S} = \text{AR}$.

- $\mathcal{K}, p \models_{\text{AR}} \exists\vec{y}.\psi(\vec{y})$ iff for every repair $(\mathcal{A}'_i)_{0\leqslant i\leqslant n}$ of $\mathcal{K}$, $\langle\mathcal{T}, (\mathcal{A}'_i)_{0\leqslant i\leqslant n}\rangle, p \models \exists\vec{y}.\psi(\vec{y})$, i.e., iff for every repair $(\mathcal{A}'_i)_{0\leqslant i\leqslant n}$ of $\mathcal{K}$, $p \leqslant n$ and $\langle\mathcal{T}, \mathcal{A}'_p\rangle \models \exists\vec{y}.\psi(\vec{y})$ because $(\mathcal{A}'_i)_{0\leqslant i\leqslant n}$ is consistent. Since the repairs of $\mathcal{K}$ are the sequences of the repairs of the $\langle\mathcal{T}, \mathcal{A}_i\rangle$, it is the case iff $p \leqslant n$ and for every repair $\mathcal{A}'_p$ of $\langle\mathcal{T}, \mathcal{A}_p\rangle$, $\langle\mathcal{T}, \mathcal{A}'_p\rangle \models \exists\vec{y}.\psi(\vec{y})$, i.e., iff $p \leqslant n$ and $\langle\mathcal{T}, \mathcal{A}_p\rangle \models_{\text{AR}} \exists\vec{y}.\psi(\vec{y})$.
- $\mathcal{K}, p \models_{\text{AR}} \phi_1 \wedge \phi_2$ iff for every repair $(\mathcal{A}'_i)_{0\leqslant i\leqslant n}$ of $\mathcal{K}$, $\langle\mathcal{T}, (\mathcal{A}'_i)_{0\leqslant i\leqslant n}\rangle, p \models \phi_1 \wedge \phi_2$, i.e., iff for every repair $(\mathcal{A}'_i)_{0\leqslant i\leqslant n}$ of $\mathcal{K}$, $\langle\mathcal{T}, (\mathcal{A}'_i)_{0\leqslant i\leqslant n}\rangle, p \models \phi_1$ and $\langle\mathcal{T}, (\mathcal{A}'_i)_{0\leqslant i\leqslant n}\rangle, p \models \phi_2$ because $(\mathcal{A}'_i)_{0\leqslant i\leqslant n}$ is consistent. It follows that $\mathcal{K}, p \models_{\text{AR}} \phi_1 \wedge \phi_2$ iff $\mathcal{K}, p \models_{\text{AR}} \phi_1$ and $\mathcal{K}, p \models_{\text{AR}} \phi_2$.
- We show all remaining relations in the same way, applying the definition of AR semantics and using the fact that TKB repairs are consistent.

The following counter-examples show that this is not the case for the other operators: $\vee, \Diamond, \Diamond^b, \Diamond^-, \mathsf{U}, \mathsf{U}^b$, and S.

- $\mathcal{K}, 0 \models_{\text{AR}} \phi_1 \vee \phi_2$ but $\mathcal{K}, 0 \not\models_{\text{AR}} \phi_1$ and $\mathcal{K}, 0 \not\models_{\text{AR}} \phi_2$:

$$\begin{aligned}\mathcal{T} &= \{A \sqsubseteq \neg B\} & \mathcal{A}_0 &= \{A(a), B(a)\}\\ \phi_1 &= A(a) & \phi_2 &= B(a)\end{aligned}$$

- $\mathcal{K}, 0 \models_{\text{AR}} \Diamond\phi_1$ (resp. $\mathcal{K}, 0 \models_{\text{AR}} \Diamond^b\phi_1$) but for every k (resp. such that $0 \leqslant k \leqslant 2$), $\mathcal{K}, k \not\models_{\text{AR}} \phi_1$:

$$\begin{aligned}\mathcal{T} &= \{A \sqsubseteq \neg B\} & \mathcal{A}_0 &= \{A(a)\}\\ \mathcal{A}_1 &= \{A(a), B(a)\} & \mathcal{A}_2 &= \{B(a)\}\\ \phi_1 &= A(a) \wedge \Circle B(a)\end{aligned}$$

- $\mathcal{K}, 0 \models_{\text{AR}} \phi_1 \mathsf{U} \phi_2$ (resp. $\mathcal{K}, 0 \models_{\text{AR}} \phi_1 \mathsf{U}^b \phi_2$) but for every k (resp. such that $0 \leqslant k \leqslant 2$), either $\mathcal{K}, k \not\models_{\text{AR}} \phi_2$ or there exists j, such that $0 \leqslant j < k$ and $\mathcal{K}, j \not\models_{\text{AR}} \phi_1$:

$$\begin{aligned}\mathcal{T} &= \{A \sqsubseteq \neg B\} & \mathcal{A}_0 &= \{A(a)\}\\ \mathcal{A}_1 &= \{A(a), B(a)\} & \mathcal{A}_2 &= \{B(a)\}\\ \phi_1 &= A(a) & \phi_2 &= B(a)\end{aligned}$$

- We can construct similar counter-examples for $\Diamond^-$ and S.

Interestingly, contrary to the brave semantics, even for general TCQs the "if" direction of Proposition 9.12 is true.

- If $\mathcal{K}, p \models_{\text{AR}} \phi_1$ or $\mathcal{K}, p \models_{\text{AR}} \phi_2$, then $\mathcal{K}, p \models_{\text{AR}} \phi_1 \vee \phi_2$.
- If there exists $k \geqslant p$ such that $\mathcal{K}, k \models_{\text{AR}} \phi_1$, then $\mathcal{K}, p \models_{\text{AR}} \Diamond\phi_1$.
- If there exists k such that $p \leqslant k \leqslant n$ and $\mathcal{K}, k \models_{\text{AR}} \phi_1$, then $\mathcal{K}, p \models_{\text{AR}} \Diamond^b\phi_1$.
- If there exists k such that $0 \leqslant k \leqslant p$ and $\mathcal{K}, k \models_{\text{AR}} \phi_1$, then $\mathcal{K}, p \models_{\text{AR}} \Diamond^-\phi_1$.
- If there exists $k \geqslant p$ such that $\mathcal{K}, k \models_{\text{AR}} \phi_2$ and for every j such that $p \leqslant j < k$, $\mathcal{K}, j \models_{\text{AR}} \phi_1$, then $\mathcal{K}, p \models_{\text{AR}} \phi_1 \mathsf{U} \phi_2$.
- If there exists k such that $p \leqslant k \leqslant n$, $\mathcal{K}, k \models_{\text{AR}} \phi_2$ and for every j such that $p \leqslant j < k$, $\mathcal{K}, j \models_{\text{AR}} \phi_1$, then $\mathcal{K}, p \models_{\text{AR}} \phi_1 \mathsf{U}^b \phi_2$.
- If there exists k such that $0 \leqslant k \leqslant p$, $\mathcal{K}, k \models_{\text{AR}} \phi_2$ and for every j such that $k < j \leqslant p$, $\mathcal{K}, j \models_{\text{AR}} \phi_1$, then $\mathcal{K}, p \models_{\text{AR}} \phi_1 \mathsf{S} \phi_2$.

It follows that even for unrestricted TCQs, combining algorithms for TCQ answering with algorithms for

AR query answering will provide *a sound approximation* of AR answers.

For brave semantics, it would be useful to characterize the queries for which this method would be correct. Indeed, for many pairs of a TBox and a query, the minimal subsets of the TKB such that the query can be mapped into them cannot be inconsistent. For instance, for DL-Lite$_\mathcal{R}$ TKBs, this is the case if no pair of predicates that may be involved at the same time point appears in an NI entailed by the TBox. Consider for instance $\mathcal{T} = \{A \sqsubseteq \neg C,\ B \sqsubseteq \neg C\}$ and $\phi = \exists x.A(x) \wedge \Diamond(\exists x.B(x) \wedge \bigcirc(\exists x.C(x)))$. For ϕ to be entailed at time point p, $\exists x.A(x)$ should hold at p, $\exists x.B(x)$ at time point $i \geqslant p$ and $\exists x.C(x)$ at $i + 1 > p$, so there cannot be a conflict between the C and the A or B timed assertions used to satisfy the different CQs.

10. Conclusions and future work

For stream reasoning handling the temporal dimension of the collected data and being resilient against errors in the data are expedient requirements. In the presence of erroneous data handling inconsistencies is even indispensable for logic-based approaches to stream reasoning. In this paper we have lifted standard inconsistency-tolerant semantics AR, IAR and brave to a temporal query answering setting that has been widely studied in the literature—namely, where the data is associated with time points and only the query language admits the use of temporal operators from LTL. We have presented complexity results and techniques to combine temporal with inconsistency-tolerant query answering over lightweight DL temporal knowledge bases suited for ontology-mediated situation recognition.

Our main contribution is a complexity analysis of the three semantics, focusing on the DLs $\mathcal{EL}_\bot$ and DL-Lite$_\mathcal{R}$, where we distinguished the cases based on whether rigid concept or role names occur in the TKB, and on whether the query contains negation. We provided general algorithms that allow us to derive the complexity of temporal inconsistency-tolerant query answering from the complexities of consistency checking and classical entailment of temporal conjunctive queries. We furthermore completed the complexity picture for the classical semantics for TCQs without negations. Indeed, for the case where the query language does not provide negation, we devised a general approach to assess the complexity by the use of the canonical model property for B(T)CQ answering and thus not only limited to a particular DL. This approach allows to derive the complexity of temporal query answering from the complexity of conjunctive queries entailment for DLs that have this canonical model property.

Encouragingly, our analysis shows that either with or without negation in the query, in most cases, inconsistency-tolerant reasoning and temporal query answering can be combined without increasing the computational complexity. Furthermore, our results show that disallowing negation in the query language results in a drop in the combined complexity of TCQ answering, and, in the case of $\mathcal{EL}_\bot$ with rigid predicates, even in the data complexity. This raises hope that ontology-based stream reasoning applications in temporal settings which are resilient against noise in the data, can be feasibly implemented and used.

As a second major contribution, we investigated two techniques useful for developing practical algorithms for inconsistency-tolerant temporal query answering. We first showed that in DL-Lite$_\mathcal{R}$, under the classical semantics and for queries without negation nor unbounded temporal operators, rigid predicates can be handled by adding a set of assertions of polynomial size to each ABox from the TKB. However, our approach that adds this set of assertions $\mathcal{R}$ to every ABox of the TKB to reduce TCQ answering with rigid predicates to TCQ answering without rigid predicates works only for BTCQ entailment under the classical semantics.

We then showed that in the case without rigid predicates and for queries without negation, TCQ answering under IAR semantics can be implemented by combining algorithms developed for TCQ answering under the classical semantics with algorithms for CQ answering under IAR semantics over atemporal KBs. Moreover, we showed that when disallowing some of the operators in the queries, the same method can be used for AR semantics, while for full TCQs without negation, it provides for a sound approximation of the AR answers. Unfortunately, this is not the case for brave semantics which are relevant for practical applications, such as recognizing highly critical situations. Thus it would be useful to characterize the queries and TBoxes for which this method is correct. Now, fully fledged practical algorithms still remain to be found for inconsistency-tolerant temporal query answering with rigid predicates.

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Appendix A. Omitted proofs

We start by defining the notions of conflicts and causes that will be used in some proofs. A *conflict* for a KB $\mathcal{K} = \langle \mathcal{T}, \mathcal{A} \rangle$ is a minimal $\mathcal{T}$-inconsistent subset of $\mathcal{A}$. A *cause* for a BCQ q w.r.t. $\mathcal{K}$ is a minimal $\mathcal{T}$-consistent subset $\mathcal{C} \subseteq \mathcal{A}$ such that $\langle \mathcal{T}, \mathcal{C} \rangle \models q$. The following definitions extend these notions to the temporal setting.

Definition A.1 (Conflicts of a TKB)**.** A *conflict* of a TKB $\mathcal{K} = \langle \mathcal{T}, (\mathcal{A}_i)_{0 \leqslant i \leqslant n} \rangle$ is a sequence of ABoxes $(\mathcal{A}'_i)_{0 \leqslant i \leqslant n}$ such that $\{(\alpha, i) \mid \alpha \in \mathcal{A}'_i, 0 \leqslant i \leqslant n\}$ is a minimal $\mathcal{T}$-inconsistent subset of $\{(\alpha, i) \mid \alpha \in \mathcal{A}_i, 0 \leqslant i \leqslant n\}$.

The conflicts of a DL-Lite$_{\mathcal{R}}$ TKB are at most binary, i.e., contain at most two timed assertions (Fact 6.11).

Definition A.2 (Causes for a BTCQ in a TKB)**.** A *cause* for a BTCQ ϕ at time point p in $\mathcal{K} = \langle \mathcal{T}, (\mathcal{A}_i)_{0 \leqslant i \leqslant n} \rangle$ is a sequence of ABoxes $(\mathcal{C}_i)_{0 \leqslant i \leqslant n}$ such that $\{(\alpha, i) \mid \alpha \in \mathcal{C}_i, 0 \leqslant i \leqslant n\}$ is a minimal $\mathcal{T}$-consistent subset of $\{(\alpha, i) \mid \alpha \in \mathcal{A}_i, 0 \leqslant i \leqslant n\}$ such that $\langle \mathcal{T}, (\mathcal{C}_i)_{0 \leqslant i \leqslant n} \rangle, p \models \phi$.

Note that a KB (resp. TKB) is consistent iff it has no conflict, and that a BCQ (resp. BTCQ) is entailed from a KB (resp. a TKB) $\mathcal{K}$ under brave semantics iff it has some cause in $\mathcal{K}$, since such a cause can be extended to a repair that entails the query.

A.1. Proofs of complexity results

Hardness-results.

Proposition 6.9. *BTCQ entailment from an $\mathcal{EL}_{\bot}$ TKB with $\mathsf{N_{RC}} \neq \emptyset$ is*

- *co*NP*-hard w.r.t. data complexity under AR and IAR semantics, and*
- Σ_2^p*-hard w.r.t. data complexity under brave semantics.*

Proof. The lower bounds for AR and IAR semantics follow from the atemporal case, so that we only have to provide the lower bound for brave semantics.

We show that the complement of brave TCQ entailment is Π_2^p-hard by reduction from QBF$_{2,\forall}$. Let $\Phi = \forall x_1 \dots x_m \exists y_1 \dots y_r \, \varphi$ be a QBF$_{2,\forall}$-formula, where $\varphi = \bigwedge_{i=0}^{h} \ell_i^0 \vee \ell_i^1 \vee \ell_i^2$ is a 3-CNF formula over the propositional variables $\{x_1, \dots, x_m, y_1, \dots, y_r\}$. Based on Φ, we define the TKB $\mathcal{K} = \langle \mathcal{T}, (\mathcal{A}_i)_{0 \leqslant i \leqslant 3h+2} \rangle$ and the TCQ ϕ as follows, where $\mathsf{N_{RC}} = \{\mathsf{T}\}$.

$$\begin{aligned}
\mathcal{T} =& \{\exists \mathsf{Pos.T} \sqsubseteq \mathsf{Sat}, \exists \mathsf{Neg.F} \sqsubseteq \mathsf{Sat}, \\
& \exists \mathsf{FromPos.Sat} \sqsubseteq \mathsf{T}, \exists \mathsf{FromNeg.Sat} \sqsubseteq \mathsf{F}, \\
& \exists \mathsf{FromY.Sat} \sqsubseteq \mathsf{T}, \mathsf{T} \sqcap \mathsf{F} \sqsubseteq \bot, \\
& \mathsf{T} \sqcap \exists \mathsf{ValY.T} \sqsubseteq \bot\} \\
\phi =& \neg \Box^b (\mathsf{NotFirst}(c) \vee \mathsf{Sat}(c) \vee \\
& \bigcirc \mathsf{Sat}(c) \vee \bigcirc\bigcirc \mathsf{Sat}(c))
\end{aligned}$$

For each clause $\ell_i^0 \vee \ell_i^1 \vee \ell_i^2$, we define the following three ABoxes $\mathcal{A}_{3i+k}$ $(0 \leqslant k \leqslant 2)$:

$$\begin{aligned}
\mathcal{A}_{3i} =& \mathcal{B} \cup \mathcal{B}_{3i} \\
\mathcal{A}_{3i+k} =& \mathcal{B} \cup \mathcal{B}_{3i+k} \cup \{\mathsf{NotFirst}(c)\}, 1 \leqslant k \leqslant 2,
\end{aligned}$$

where

$$\begin{aligned}\mathcal{B} =&\{\mathsf{T}(x_j), \mathsf{F}(x_j) \mid 1 \leqslant j \leqslant m\} \cup\\ &\{\mathsf{ValY}(y_j, \neg y_j), \mid 1 \leqslant j \leqslant r\}\\ \mathcal{B}_{3i+k} =&\{\mathsf{Pos}(c, x_j), \mathsf{FromPos}(x_j, c)\} \text{ if } \ell_i^k = x_j\\ \mathcal{B}_{3i+k} =&\{\mathsf{Neg}(c, x_j), \mathsf{FromNeg}(x_j, c)\} \text{ if } \ell_i^k = \neg x_j\\ \mathcal{B}_{3i+k} =&\{\mathsf{FromY}(y_j, c)\} \text{ if } \ell_i^k = y_j\\ \mathcal{B}_{3i+k} =&\{\mathsf{FromY}(\neg y_j, c)\} \text{ if } \ell_i^k = \neg y_j.\end{aligned}$$

We show that Φ is valid iff $\mathcal{K}, 0 \not\models_{\text{brave}} \phi$.

The repairs of $\mathcal{K}$ correspond to the valuations of the x_j. Indeed, since T is rigid and disjoint from F, each pair of timed assertions $\{(\mathsf{T}(x_j), i_1), (\mathsf{N}(x_j), i_2)\}$ is inconsistent, so every x_j is such that a repair of $\mathcal{K}$ contains either $(\mathsf{T}(x_j), i)$ for every i, or $(\mathsf{F}(x_j), i)$ for every i. For each repair $\mathcal{A}' = (\mathcal{A}'_i)_{0\leqslant i\leqslant 3h+2}$ of $\mathcal{K}$, we denote by $\nu_X^{\mathcal{A}'}$ the valuation of the x_j defined by $\nu_X^{\mathcal{A}'}(x_j) = \mathsf{true}$ if $\mathsf{T}(x_j) \in \mathcal{A}'_1$. Correspondingly, for every valuation ν_X of the x_j, we denote by $\mathcal{A}^{\nu_X} = (\mathcal{A}_i^{\nu_X})_{0\leqslant i\leqslant 3h+2}$ the repair of $\mathcal{K}$ defined by $\mathsf{T}(x_j) \in \mathcal{A}_i^{\nu_X}$ for every i if $\nu_X(x_j) = \mathsf{true}$.

Assume that Φ is valid and let $\mathcal{A}' = (\mathcal{A}'_i)_{0\leqslant i\leqslant 3h+2}$ be a repair of $\mathcal{K}$. Since Φ is valid, then there exists a valuation ν_Y of the y_j such that $\varphi[x_j \leftarrow \nu_X^{\mathcal{A}'}(x_j)]$ is satisfied by ν_Y. Let $\mathcal{J} = (\mathcal{I}_i)_{i\geqslant 0}$ be a model of $\langle\mathcal{T}, (\mathcal{A}'_i)_{0\leqslant i\leqslant 3h+2}\rangle$ that respects rigid predicates and such that for every i,

- $y_j^{\mathcal{I}_i} \in \mathsf{T}^{\mathcal{I}_i}$ iff $\nu_Y(y_j) = \mathsf{true}$,
- $\neg y_j^{\mathcal{I}_i} \in \mathsf{T}^{\mathcal{I}_i}$ iff $\nu_Y(y_j) = \mathsf{false}$, and
- if there exists some d such that $(d^{\mathcal{I}_i}, c^{\mathcal{I}_i}) \in \mathsf{FromY}^{\mathcal{I}_i}$ and $d^{\mathcal{I}_i} \in \mathsf{T}^{\mathcal{I}_i}$, then $c^{\mathcal{I}_i} \in \mathsf{Sat}^{\mathcal{I}_i}$.

One can verify that such a model always exists. First, because the role ValY links only individuals of the type y_j and $\neg y_j$, and we only assign T to one of them, these additional constraints respect the TBox axiom $\mathsf{T} \sqcap \exists\mathsf{ValY}.\mathsf{T} \sqsubseteq \bot$. Second, the assignment of c to Sat respects $\exists\mathsf{FormY}.\mathsf{Sat} \sqsubseteq \mathsf{T}$ by construction.

It is easy to see that $\mathcal{J}, 0 \models \Box^b(\mathsf{NotFirst}(c) \vee \mathsf{Sat}(c) \vee \bigcirc\mathsf{Sat}(c) \vee \bigcirc\bigcirc\mathsf{Sat}(c))$. Indeed, at each time point $p \in [\![0, 3h+2]\!]$, either $\mathsf{NotFirst}(c)$ is true, or $p = 3i$ and we show that $\mathsf{Sat}(c)$ is true at time point $3i+k$, where ℓ_i^k is the first literal of the clause $\ell_i^0 \vee \ell_i^1 \vee \ell_i^2$ satisfied by $\nu_X^{\mathcal{A}'} \cup \nu_Y$.

- If $\ell_i^k = x_j$, then $\nu_X^{\mathcal{A}'}(x_j) = \mathsf{true}$. Thus, by construction, $\mathsf{T}(x_j) \in \mathcal{A}'_{3i+k}$, and therefore $\mathcal{J}, 3i+k \models \mathsf{T}(x_j)$. Moreover, since $\mathcal{J}, 3i+k \models \mathsf{Pos}(c, x_j)$, also $\mathcal{J}, 3i+k \models \mathsf{Sat}(c)$, because $\mathcal{J}$ is a model of $\mathcal{T}$.
- If $\ell_i^k = \neg x_j$, then $\nu_X^{\mathcal{A}'}(x_j) = \mathsf{false}$. Thus, by construction, $\mathsf{F}(x_j) \in \mathcal{A}'_{3i+k}$, and therefore $\mathcal{J}, 3i+k \models \mathsf{F}(x_j)$. Moreover, since $\mathcal{J}, 3i+k \models \mathsf{Neg}(c, x_j)$, we obtain $\mathcal{J}, 3i+k \models \mathsf{Sat}(c)$ because $\mathcal{J}$ is a model of $\mathcal{T}$.
- If $\ell_i^k = y_j$, then $\nu_Y(y_j) = \mathsf{true}$. Thus $\mathcal{J}, 3i+k \models \mathsf{T}(y_j)$, and since $\mathcal{J}, 3i+k \models \mathsf{FromY}(y_j, c)$, by construction of $\mathcal{J}$, it follows that $\mathcal{J}, 3i+k \models \mathsf{Sat}(c)$.
- If $\ell_i^k = \neg y_j$, then $\nu_Y(y_j) = \mathsf{false}$. Thus $\mathcal{J}, 3i+k \models \mathsf{T}(\neg y_j)$, and since $\mathcal{J}, 3i+k \models \mathsf{FromY}(\neg y_j, c)$, by construction of $\mathcal{J}$, it follows that $\mathcal{J}, 3i+k \models \mathsf{Sat}(c)$.

It follows that $\mathcal{J}, 0 \not\models \phi$, so $\langle\mathcal{T}, (\mathcal{A}'_i)_{0\leqslant i\leqslant 3h+2}\rangle, 0 \not\models \phi$. Hence, $\mathcal{K}, 0 \not\models_{\text{brave}} \phi$.

In the other direction, assume that $\mathcal{K}, 0 \not\models_{\text{brave}} \phi$, and let ν_X be a valuation of the x_j. Since $(\mathcal{A}_i^{\nu_X})_{0\leqslant i\leqslant 3h+2}$ is a repair of $\mathcal{K}$, $\langle\mathcal{T}, (\mathcal{A}_i^{\nu_X})_{0\leqslant i\leqslant 3h+2}\rangle, 0 \not\models \phi$, so there exists a model $\mathcal{J} = (\mathcal{I}_i)_{i\geqslant 0}$ of $\langle\mathcal{T}, (\mathcal{A}_i^{\nu_X})_{0\leqslant i\leqslant 3h+2}\rangle$ that respects rigid predicates and is such that $\mathcal{J}, 0 \not\models \phi$, i.e., $\mathcal{J}, 0 \models \Box^b(\mathsf{NotFirst}(c) \vee \mathsf{Sat}(c) \vee \bigcirc\mathsf{Sat}(c) \vee \bigcirc\bigcirc\mathsf{Sat}(c))$. Let ν_Y be the (partial) valuation of the y_j defined as follows: $\nu_Y(y_j) = \mathsf{true}$ if there exists k such that $\mathcal{J}, k \models \mathsf{T}(y_j)$, and $\nu_Y(y_j) = \mathsf{false}$ if there exists k such that $\mathcal{J}, k \models \mathsf{T}(\neg y_j)$. The valuation ν_Y is well defined because $\mathcal{J}$ is a model of $\mathcal{T}$ and respects rigid predicates. Therefore, if $\mathcal{J}, k \models \mathsf{T}(y_j)$ for some k, then $\mathcal{J}, k \models \mathsf{T}(y_j)$ for every k, and $\mathcal{J}, k \not\models \mathsf{T}(\neg y_j)$. Otherwise, we would have $\mathcal{J}, k \models \mathsf{T} \sqcap \exists\mathsf{ValY}.\mathsf{T}(y_j)$ which contradicts our TBox axioms.

Since $\mathcal{J}, 0 \models \Box^b(\mathsf{NotFirst}(c) \vee \mathsf{Sat}(c) \vee \bigcirc\mathsf{Sat}(c) \vee \bigcirc\bigcirc\mathsf{Sat}(c))$, for every clause $\ell_i^0 \vee \ell_i^1 \vee \ell_i^2$, we have that $\mathcal{J}, 3i+k \models \mathsf{Sat}(c)$ for some $k \in [\![0, 2]\!]$. We then show that $\nu_X \cup \nu_Y$ satisfies ℓ_i^k.

- If $\ell_i^k = x_j$, since $\mathcal{J}, 3i+k \models \mathsf{Sat}(c)$, $\mathcal{J}, 3i+k \models \mathsf{FromPos}(x_j, c)$ and $\mathcal{J}$ respects $\exists\mathsf{FromPos}.\mathsf{Sat} \sqsubseteq \mathsf{T}$, then $\mathcal{J}, 3i+k \models \mathsf{T}(x_j)$. It follows that $(\mathsf{T}(x_j), k) \in (\mathcal{A}_i^{\nu_X})_{0\leqslant i\leqslant 3h+2}$ for every k (otherwise, by maximality of repairs, $(\mathsf{F}(x_j), k) \in (\mathcal{A}_i^{\nu_X})_{0\leqslant i\leqslant 3h+2}$ and $\mathcal{J}$ assigns x_j to T and F at some time point). Hence, $\nu_X(x_j) = \mathsf{true}$.
- If $\ell_i^k = \neg x_j$, since $\mathcal{J}, 3i+k \models \mathsf{Sat}(c)$, $\mathcal{J}, 3i+k \models \mathsf{FromNeg}(x_j, c)$ and $\mathcal{J}$ respects $\exists\mathsf{FromNeg}.\mathsf{Sat} \sqsubseteq \mathsf{F}$, then $\mathcal{J}, 3i+k \models \mathsf{F}(x_j)$. It follows that $(\mathsf{F}(x_j), k) \in (\mathcal{A}_i^{\nu_X})_{0\leqslant i\leqslant 3h+2}$ for every k (otherwise $(\mathsf{T}(x_j), k) \in (\mathcal{A}_i^{\nu_X})_{0\leqslant i\leqslant 3h+2}$ and $\mathcal{J}$ assigns x_j to T and F at some time point). Hence $\nu_X(x_j) = \mathsf{false}$.
- If $\ell_i^k = y_j$, since $\mathcal{J}, 3i+k \models \mathsf{Sat}(c)$, $\mathcal{J}, 3i+k \models \mathsf{FromY}(y_j, c)$ and $\mathcal{J}$ respects $\exists\mathsf{FromY}.\mathsf{Sat} \sqsubseteq \mathsf{T}$, then $\mathcal{J}, 3i+k \models \mathsf{T}(y_j)$, so $\nu_Y(x_j) = \mathsf{true}$.

– If $\ell_i^k = \neg y_j$, since $\mathcal{J}, 3i+k \models \mathsf{Sat}(c)$, $\mathcal{J}, 3i+k \models \mathsf{FromY}(\neg y_j, c)$ and $\mathcal{J}$ respects $\exists\mathsf{FromY}.\mathsf{Sat} \sqsubseteq \mathsf{T}$, then $\mathcal{J}, 3i+k \models \mathsf{T}(\neg y_j)$, so $\nu_Y(x_j) = \mathsf{false}$.

It follows that $\nu_X \cup \nu_Y$ satisfies every clause $\ell_i^0 \vee \ell_i^1 \vee \ell_i^2$. We have thus shown that $\varphi[x_j \leftarrow \nu_X(x_j), y_j \leftarrow \nu_Y(y_j)]$ evaluates to true, and that Φ is valid. $\square$

Justification structures for brave entailment of BTCQs without negation in the case $\mathsf{N_{RC}} = \mathsf{N_{RR}} = \emptyset$. We prove that if $\mathsf{N_{RC}} = \mathsf{N_{RR}} = \emptyset$ and ϕ is a BTCQ without negation, then $\mathcal{K}, p \models_{\text{brave}} \phi$ iff there is a correct brave-justification structure J for ϕ in $\mathcal{K}$ that justifies ϕ at time point p. We prove both directions in separate lemmas.

Lemma A.3. *If* $\mathsf{N_{RC}} = \mathsf{N_{RR}} = \emptyset$ *and there is a correct brave-justification structure J for ϕ in $\mathcal{K}$ that justifies ϕ at time point p, then $\mathcal{K}, p \models_{brave} \phi$.*

Proof. In order to show $\mathcal{K}, p \models_{\text{brave}} \phi$, we determine a cause $(\mathcal{C}_i)_{0 \leqslant i \leqslant n}$ for ϕ. To do this, we first select a sequence of tuples from J as follows:

1. The tuple $(p, L^p_{\text{now}}, F^p_{\text{now}}, F^p_{\text{prev}}, F^p_{\text{next}})$ is such that $\phi \in F^p_{\text{now}}$.
2. If the tuple $(i, L^i_{\text{now}}, F^i_{\text{now}}, F^i_{\text{prev}}, F^i_{\text{next}})$ was selected and $0 < i \leqslant p$, then select a tuple of the form $(i-1, L^{i-1}_{\text{now}}, F^{i-1}_{\text{now}}, F^{i-1}_{\text{prev}}, F^{i-1}_{\text{next}})$, where $F^{i-1}_{\text{now}} = F^i_{\text{prev}}$ and $F^{i-1}_{\text{next}} = F^i_{\text{now}}$.
3. If the tuple $(i, L^i_{\text{now}}, F^i_{\text{now}}, F^i_{\text{prev}}, F^i_{\text{next}})$ was selected and $p \leqslant i < n$, then select a tuple of the form $(i+1, L^{i+1}_{\text{now}}, F^{i+1}_{\text{now}}, F^{i+1}_{\text{prev}}, F^{i+1}_{\text{next}})$, where $F^{i+1}_{\text{now}} = F^i_{\text{next}}$ and $F^{i+1}_{\text{prev}} = F^i_{\text{now}}$.

Because J is correct and justifies ϕ at time point p, such a sequence can always be selected. Based on this sequence, we construct a sequence of ABoxes $(\mathcal{C}_i)_{0 \leqslant i \leqslant n}$. For this, we take for each of the tuples $(i, L^i_{\text{now}}, F^i_{\text{now}}, F^i_{\text{prev}}, F^i_{\text{next}})$ a cause $\mathcal{C}_i \subseteq \mathcal{A}_i$ that entails $\bigwedge_{q \in L^i_{\text{now}}} q$. Such a cause exists because $\langle \mathcal{T}, \mathcal{A}_i \rangle \models_{\text{brave}} \bigwedge_{q \in L^i_{\text{now}}} q$ by Condition 1. Since each $\mathcal{C}_i$ is consistent and rigid predicates are not allowed, the TKB $\langle \mathcal{T}, (\mathcal{C}_i)_{0 \leqslant i \leqslant n} \rangle$ is consistent.

We prove that $\langle \mathcal{T}, (\mathcal{C}_i)_{0 \leqslant i \leqslant n} \rangle, p \models \phi$, by proving that $\langle \mathcal{T}, (\mathcal{C}_i)_{0 \leqslant i \leqslant n} \rangle, p \models F^p_{\text{now}}$. To do this, we consider the sets of LTL formulas $F^{i,d}_{\text{now}} = \{\psi \mid \psi \in F^i_{\text{now}}, \mathsf{degree}(\psi) \leqslant d\}$, where $\mathsf{degree}(\psi)$ is the maximal number of nested LTL operators in ψ, and prove by induction on d that for all $0 \leqslant i \leqslant n$ and for all $\psi \in F^{i,d}_{\text{now}}$, we have $\langle \mathcal{T}, (\mathcal{C}_i)_{0 \leqslant i \leqslant n} \rangle, i \models \psi$, i.e., $\langle \mathcal{T}, (\mathcal{C}_i)_{0 \leqslant i \leqslant n} \rangle, i \models F^{i,d}_{\text{now}}$.

For $d = 0$, $F^{i,0}_{\text{now}}$ contains only conjunctive queries of the form $\exists \vec{y} \varphi(\vec{y})$. Since for every $\psi \in BCQ(\phi)$, if $F^i_{\text{now}} \models \psi$ then $\psi \in L^i_{\text{now}}$ (Condition 4), $F^{i,0}_{\text{now}} \subseteq L^i_{\text{now}}$. Then, since $\langle \mathcal{T}, \mathcal{C}_i \rangle \models \bigwedge_{q \in L^i_{\text{now}}} q$, it follows that $\langle \mathcal{T}, (\mathcal{C}_i)_{0 \leqslant i \leqslant n} \rangle, i \models F^{i,0}_{\text{now}}$.

Assume that for all $0 \leqslant i \leqslant n$, $\langle \mathcal{T}, (\mathcal{C}_i)_{0 \leqslant i \leqslant n} \rangle, i \models F^{i,d}_{\text{now}}$. Let $\psi \in F^{i,d+1}_{\text{now}}$ for some $0 \leqslant i \leqslant n$. If $\psi \in F^{i,d}_{\text{now}}$, then $\langle \mathcal{T}, (\mathcal{C}_i)_{0 \leqslant i \leqslant n} \rangle, i \models \psi$. Otherwise, $\mathsf{degree}(\psi) = d+1$ and we distinguish the cases based on the syntactical form of ψ.

– $\psi = \psi_1 \wedge \psi_2$, where $\mathsf{degree}(\psi_1) \leqslant d$, $\mathsf{degree}(\psi_2) \leqslant d$. Since $\psi \in F^i_{\text{now}}$, then $F^i_{\text{now}} \models \psi_1$ and $F^i_{\text{now}} \models \psi_2$, so by Condition 5, $\psi_1 \in F^i_{\text{now}}$ and $\psi_2 \in F^i_{\text{now}}$. It follows that $\psi_1 \in F^{i,d}_{\text{now}}$ and $\psi_2 \in F^{i,d}_{\text{now}}$, so $\langle \mathcal{T}, (\mathcal{C}_i)_{0 \leqslant i \leqslant n} \rangle, i \models \psi_1$ and $\langle \mathcal{T}, (\mathcal{C}_i)_{0 \leqslant i \leqslant n} \rangle, i \models \psi_2$. Hence $\langle \mathcal{T}, (\mathcal{C}_i)_{0 \leqslant i \leqslant n} \rangle, i \models \psi_1 \wedge \psi_2$.
– $\psi = \psi_1 \vee \psi_2$, where $\mathsf{degree}(\psi_1) \leqslant d$, $\mathsf{degree}(\psi_2) \leqslant d$. Since $\psi \in F^i_{\text{now}}$, then by Condition 7 either $\psi_1 \in F^i_{\text{now}}$ or $\psi_2 \in F^i_{\text{now}}$. It follows that $\psi_1 \in F^{i,d}_{\text{now}}$ or $\psi_2 \in F^{i,d}_{\text{now}}$, so that $\langle \mathcal{T}, (\mathcal{C}_i)_{0 \leqslant i \leqslant n} \rangle, i \models \psi_1$ or $\langle \mathcal{T}, (\mathcal{C}_i)_{0 \leqslant i \leqslant n} \rangle, i \models \psi_2$. Hence, $\langle \mathcal{T}, (\mathcal{C}_i)_{0 \leqslant i \leqslant n} \rangle, i \models \psi_1 \vee \psi_2$.
– $\psi = \bigcirc \psi_1$, where $\mathsf{degree}(\psi_1) \leqslant d$. By Condition 8, either $i < n$, or $i = n$ and $\langle \emptyset, (\emptyset)_{0 \leqslant i \leqslant n} \rangle, n+1 \not\models \Box \varphi$. In the latter case, note that the canonical models of $\langle \mathcal{T}, (\mathcal{C}_i)_{0 \leqslant i \leqslant n} \rangle$ and $\langle \emptyset, (\emptyset)_{0 \leqslant i \leqslant n} \rangle$ coincide after n (empty interpretations), and since ψ_1 does not contain any past operators, $\langle \mathcal{T}, (\mathcal{C}_i)_{0 \leqslant i \leqslant n} \rangle, n+1 \models \Box \psi_1$ is a direct consequence of $\langle \emptyset, (\emptyset)_{0 \leqslant i \leqslant n} \rangle, n+1 \models \Box \psi_1$. Then ψ_1 is true at any time point $j > n$ and in particular, $\langle \mathcal{T}, (\mathcal{C}_i)_{0 \leqslant i \leqslant n} \rangle, n \models \bigcirc \psi_1$.
In the former case, since $\bigcirc \psi_1 \in F^i_{\text{now}} = F^{i+1}_{\text{prev}}$, we have that $\bigwedge_{q \in L^{i+1}_{\text{now}}} q \wedge \bigcirc^-(\bigwedge_{\chi \in F^{i+1}_{\text{prev}}} \chi) \wedge \bigcirc(\bigwedge_{\chi \in F^{i+1}_{\text{next}}} \chi) \models \bigcirc^- \bigcirc \psi_1 \models \psi_1$, so by Condition 6, $\psi_1 \in F^{i+1}_{\text{now}}$. Hence, $\psi_1 \in F^{i+1,d}_{\text{now}}$, $\langle \mathcal{T}, (\mathcal{C}_i)_{0 \leqslant i \leqslant n} \rangle, i+1 \models \psi_1$, and $\langle \mathcal{T}, (\mathcal{C}_i)_{0 \leqslant i \leqslant n} \rangle, i \models \bigcirc \psi_1$.
– $\psi = \bigcirc^- \psi_1$, where $\mathsf{degree}(\psi_1) \leqslant d$. This case is similar to $\bigcirc$.
– $\psi = \bullet^b \psi_1$, where $\mathsf{degree}(\psi_1) \leqslant d$. If $i < n$, since $\bullet^b \psi_1 \in F^i_{\text{now}} = F^{i+1}_{\text{prev}}$, we have that $\bigwedge_{q \in L^{i+1}_{\text{now}}} q \wedge \bigcirc^-(\bigwedge_{\chi \in F^{i+1}_{\text{prev}}} \chi) \wedge \bigcirc(\bigwedge_{\chi \in F^{i+1}_{\text{next}}} \chi) \models \bigcirc^- \bullet^b \psi_1 \models \psi_1$, so that by Condition 6, $\psi_1 \in F^{i+1}_{\text{now}}$. Hence $\psi_1 \in F^{i+1,d}_{\text{now}}$ and $\langle \mathcal{T}, (\mathcal{C}_i)_{0 \leqslant i \leqslant n} \rangle, i+1 \models \psi_1$, which implies $\langle \mathcal{T}, (\mathcal{C}_i)_{0 \leqslant i \leqslant n} \rangle, i \models \bullet \psi_1$. Otherwise, $i = n$, and $\langle \mathcal{T}, (\mathcal{C}_i)_{0 \leqslant i \leqslant n} \rangle, n \models \bullet^b \psi_1$ by definition of $\bullet^b$.
– $\psi = \bullet^- \psi_1$ where $\mathsf{degree}(\psi_1) \leqslant d$. This case is similar to $\bullet^b$.
– $\psi = \Box \psi_1$, where $\mathsf{degree}(\psi_1) \leqslant d$. By Condition 7, $\langle \emptyset, (\emptyset)_{0 \leqslant i \leqslant n} \rangle, n+1 \models \Box \psi_1$.

We show that $\langle \mathcal{T}, (\mathcal{C}_i)_{0\leqslant i\leqslant n}\rangle, i \models \Box\psi_1$ by descending induction on i.
For $i = n$, note that the canonical models of $\langle \mathcal{T}, (\mathcal{C}_i)_{0\leqslant i\leqslant n}\rangle$ and $\langle \emptyset, (\emptyset)_{0\leqslant i\leqslant n}\rangle$ coincide after n (empty interpretations), and since ψ_1 does not contain any past operators,
$\langle \mathcal{T}, (\mathcal{C}_i)_{0\leqslant i\leqslant n}\rangle, n+1 \models \Box\psi_1$ is a direct consequence of $\langle \emptyset, (\emptyset)_{0\leqslant i\leqslant n}\rangle, n+1 \models \Box\psi_1$. Then $\langle \mathcal{T}, (\mathcal{C}_i)_{0\leqslant i\leqslant n}\rangle, n \models \Box\psi_1$ iff $\langle \mathcal{T}, (\mathcal{C}_i)_{0\leqslant i\leqslant n}\rangle, n \models \psi_1$, that is iff $\psi_1 \in F^{n,d}_{\text{now}}$ by induction. This is the case by Condition 5.
For $i < n$, we assume by inductive hypothesis that if $\Box\psi_1 \in F^{i+1}_{\text{now}}$, then $\langle \mathcal{T}, (\mathcal{C}_i)_{0\leqslant i\leqslant n}\rangle, i+1 \models \Box\psi_1$. Since $\Box\psi_1 \in F^i_{\text{now}} = F^{i+1}_{\text{prev}}$, we have that $\bigwedge_{q\in L^{i+1}_{\text{now}}} q \wedge \bigcirc^-(\bigwedge_{\chi\in F^{i+1}_{\text{prev}}}\chi) \wedge \bigcirc(\bigwedge_{\chi\in F^{i+1}_{\text{next}}}\chi) \models \bigcirc^-\Box\psi_1 \models \Box\psi_1$, so by Condition 6, $\Box\psi_1 \in F^{i+1}_{\text{now}}$, and by assumption $\langle \mathcal{T}, (\mathcal{C}_i)_{0\leqslant i\leqslant n}\rangle, i+1 \models \Box\psi_1$. Moreover, since $\Box\psi_1 \in F^i_{\text{now}}$, then $F^i_{\text{now}} \models \psi_1$, and $\psi_1 \in F^i_{\text{now}}$ by Condition 5. Hence, $\psi_1 \in F^{i,d}_{\text{now}}$ and $\langle \mathcal{T}, (\mathcal{C}_i)_{0\leqslant i\leqslant n}\rangle, i \models \psi_1$.
It follows that $\langle \mathcal{T}, (\mathcal{C}_i)_{0\leqslant i\leqslant n}\rangle, i \models \Box\psi_1$.

- $\psi = \Box^b\psi_1$ where $\mathsf{degree}(\psi_1) \leqslant d$. We show that $\langle \mathcal{T}, (\mathcal{C}_i)_{0\leqslant i\leqslant n}\rangle, i \models \Box^b\psi_1$ by descending induction on i.
For $i = n$, if $\Box^b\psi_1 \in F^n_{\text{now}}$, then $\psi_1 \in F^n_{\text{now}}$ by Condition 8, and therefore $\psi_1 \in F^{n,d}_{\text{now}}$ and $\langle \mathcal{T}, (\mathcal{C}_i)_{0\leqslant i\leqslant n}\rangle, n \models \psi_1$. As a consequence, we obtain $\langle \mathcal{T}, (\mathcal{C}_i)_{0\leqslant i\leqslant n}\rangle, n \models \Box^b\psi_1$.
For $i < n$, we assume by inductive hypothesis that if $\Box^b\psi_1 \in F^{i+1}_{\text{now}}$, then $\langle \mathcal{T}, (\mathcal{C}_i)_{0\leqslant i\leqslant n}\rangle, i+1 \models \Box^b\psi_1$. Then, since $\Box^b\psi_1 \in F^i_{\text{now}} = F^{i+1}_{\text{prev}}$, we have that $\bigwedge_{q\in L^{i+1}_{\text{now}}} q \wedge \bigcirc^-(\bigwedge_{\chi\in F^{i+1}_{\text{prev}}}\chi) \wedge \bigcirc(\bigwedge_{\chi\in F^{i+1}_{\text{next}}}\chi) \models \bigcirc^-\Box^b\psi_1 \models \Box^b\psi_1$. Therefore, by Condition 6, $\Box^b\psi_1 \in F^{i+1}_{\text{now}}$, and by assumption $\langle \mathcal{T}, (\mathcal{C}_i)_{0\leqslant i\leqslant n}\rangle, i+1 \models \Box^b\psi_1$. Moreover, since $\Box^b\psi_1 \in F^i_{\text{now}}$, then $F^i_{\text{now}} \models \psi_1$, and $\psi_1 \in F^i_{\text{now}}$ by Condition 5. Hence, $\psi_1 \in F^{i,d}_{\text{now}}$ and $\langle \mathcal{T}, (\mathcal{C}_i)_{0\leqslant i\leqslant n}\rangle, i \models \psi_1$. It follows that $\langle \mathcal{T}, (\mathcal{C}_i)_{0\leqslant i\leqslant n}\rangle, i \models \Box^b\psi_1$.
- $\psi = \Box^-\psi_1$, where $\mathsf{degree}(\psi_1) \leqslant d$. This case is similar to $\Box^b$.
- $\psi = \Diamond\psi_1$, where $\mathsf{degree}(\psi_1) \leqslant d$. We prove $\langle \mathcal{T}, (\mathcal{C}_i)_{0\leqslant i\leqslant n}\rangle, i \models \Diamond\psi_1$ by descending induction on i.
For $i = n$, if $\Diamond\psi_1 \in F^n_{\text{now}}$, then $\psi_1 \in F^n_{\text{now}}$ or $\langle \emptyset, (\emptyset)_{0\leqslant i\leqslant n}\rangle, n+1 \not\models \Box\psi_1$ by Condition 8.
In the former case, $\psi_1 \in F^{n,d}_{\text{now}}$ and $\langle \mathcal{T}, (\mathcal{C}_i)_{0\leqslant i\leqslant n}\rangle, n \models \psi_1$, which implies that $\langle \mathcal{T}, (\mathcal{C}_i)_{0\leqslant i\leqslant n}\rangle, n \models \Diamond\psi_1$.
In the latter case we can show as in the proof for $\bigcirc$ that $\langle \mathcal{T}, (\mathcal{C}_i)_{0\leqslant i\leqslant n}\rangle, n+1 \models \Box\psi_1$, which implies that $\langle \mathcal{T}, (\mathcal{C}_i)_{0\leqslant i\leqslant n}\rangle, n+1 \models \Diamond\psi_1$.
For $i < n$, we assume by inductive hypothesis that if $\Diamond\psi_1 \in F^{i+1}_{\text{now}}$, then $\langle \mathcal{T}, (\mathcal{C}_i)_{0\leqslant i\leqslant n}\rangle, i+1 \models \Diamond\psi_1$. Since $\Diamond\psi_1 \in F^i_{\text{now}}$, by Condition 7, either (i) $\psi_1 \in F^i_{\text{now}}$, $\psi_1 \in F^{i,d}_{\text{now}}$ and $\langle \mathcal{T}, (\mathcal{C}_i)_{0\leqslant i\leqslant n}\rangle, i \models \psi_1$, and therefore $\langle \mathcal{T}, (\mathcal{C}_i)_{0\leqslant i\leqslant n}\rangle, i \models \Diamond\psi_1$, or (ii) $\Diamond\psi_1 \in F^i_{\text{next}} = F^{i+1}_{\text{now}}$, and by assumption $\langle \mathcal{T}, (\mathcal{C}_i)_{0\leqslant i\leqslant n}\rangle, i+1 \models \Diamond\psi_1$. It follows that $\langle \mathcal{T}, (\mathcal{C}_i)_{0\leqslant i\leqslant n}\rangle, i \models \Diamond\psi_1$.
- $\psi = \Diamond^b\psi_1$, where $\mathsf{degree}(\psi_1) \leqslant d$. This case is similar as for $\Diamond\psi_1$.
- $\psi = \Diamond^-\psi_1$, where $\mathsf{degree}(\psi_1) \leqslant d$. This case is similar to $\Diamond$.
- $\psi = \psi_1 \mathsf{U}\, \psi_2$ where $\mathsf{degree}(\psi_1) \leqslant d$, $\mathsf{degree}(\psi_2) \leqslant d$. We show that $\langle \mathcal{T}, (\mathcal{C}_i)_{0\leqslant i\leqslant n}\rangle, i \models \psi_1 \mathsf{U}\, \psi_2$ by descending induction on i.
For $i = n$, if $\psi_1 \mathsf{U}\, \psi_2 \in F^n_{\text{now}}$, then $\psi_2 \in F^n_{\text{now}}$, or $\langle \emptyset, (\emptyset)_{0\leqslant i\leqslant n}\rangle, n \not\models \Box\psi_2$ by Condition 8.
In the former case, $\psi_2 \in F^{n,d}_{\text{now}}$ and $\langle \mathcal{T}, (\mathcal{C}_i)_{0\leqslant i\leqslant n}\rangle, n \models \psi_2$, which implies that $\langle \mathcal{T}, (\mathcal{C}_i)_{0\leqslant i\leqslant n}\rangle, n \models \psi_1 \mathsf{U}\, \psi_2$.
In the latter case, we can show as in the proof for $\bigcirc$ that $\langle \mathcal{T}, (\mathcal{C}_i)_{0\leqslant i\leqslant n}\rangle, n+1 \models \Box\psi_2$, which implies that $\langle \mathcal{T}, (\mathcal{C}_i)_{0\leqslant i\leqslant n}\rangle, n+1 \models \psi_1 \mathsf{U}\, \psi_2$. Then $\langle \mathcal{T}, (\mathcal{C}_i)_{0\leqslant i\leqslant n}\rangle, n \models \Box\psi_1 \mathsf{U}\, \psi_2$ iff $\langle \mathcal{T}, (\mathcal{C}_i)_{0\leqslant i\leqslant n}\rangle, n \models \psi_1$ or $\langle \mathcal{T}, (\mathcal{C}_i)_{0\leqslant i\leqslant n}\rangle, n \models \psi_2$, that is iff $\psi_1 \in F^{n,d}_{\text{now}}$ or $\psi_1 \in F^{n,d}_{\text{now}}$ by induction. This is the case by Condition 5.
For $i < n$, we assume by inductive hypothesis that if $\psi_1 \mathsf{U}\, \psi_2 \in F^{i+1}_{\text{now}}$, then $\langle \mathcal{T}, (\mathcal{C}_i)_{0\leqslant i\leqslant n}\rangle, i+1 \models \psi_1 \mathsf{U}\, \psi_2$. Then, since $\psi_1 \mathsf{U}\, \psi_2 \in F^i_{\text{now}}$, by Condition 7, either (i) $\psi_2 \in F^i_{\text{now}}$, $\psi_2 \in F^{i,d}_{\text{now}}$ and $\langle \mathcal{T}, (\mathcal{C}_i)_{0\leqslant i\leqslant n}\rangle, i \models \psi_2$, which in turn implies $\langle \mathcal{T}, (\mathcal{C}_i)_{0\leqslant i\leqslant n}\rangle, i \models \psi_1 \mathsf{U}\, \psi_2$, or (ii) $\psi_1 \in F^i_{\text{now}}$, $\psi_1 \in F^{i,d}_{\text{now}}$, which implies $\langle \mathcal{T}, (\mathcal{C}_i)_{0\leqslant i\leqslant n}\rangle, i \models \psi_1$, and $\psi_1 \mathsf{U}\, \psi_2 \in F^i_{\text{next}} = F^{i+1}_{\text{now}}$. Therefore, by assumption we obtain $\langle \mathcal{T}, (\mathcal{C}_i)_{0\leqslant i\leqslant n}\rangle, i+1 \models \psi_1 \mathsf{U}\, \psi_2$, and $\langle \mathcal{T}, (\mathcal{C}_i)_{0\leqslant i\leqslant n}\rangle, i \models \psi_1 \mathsf{U}\, \psi_2$.
- $\psi = \psi_1 \mathsf{U}^b\, \psi_2$, where $\mathsf{degree}(\psi_1) \leqslant d$, $\mathsf{degree}(\psi_2) \leqslant d$. This case can be shown in the same way as for U.
- $\psi = \psi_1 \mathsf{S}\, \psi_2$, where $\mathsf{degree}(\psi_1) \leqslant d$, $\mathsf{degree}(\psi_2) \leqslant d$. This case is similar to U.

$\Box$

Lemma A.4. *If $\mathsf{N}_{\mathsf{RC}} = \mathsf{N}_{\mathsf{RR}} = \emptyset$ and $\mathcal{K}, p \models_{brave} \phi$, then there is a brave-justification structure for ϕ in $\mathcal{K}$ that is correct and justifies ϕ at time point p.*

Proof. Assume $\mathcal{K}, p \models_{\text{brave}} \phi$. Then there exists a TKB $\mathcal{K}' = \langle \mathcal{T}, (\mathcal{C}_i)_{0\leqslant i\leqslant n}\rangle$, such that $\mathcal{C}_i \subseteq \mathcal{A}_i$ and $\mathcal{K}'$ is consistent and $\mathcal{K}', p \models \phi$. Based on $\mathcal{K}'$, we construct a brave-justification structure J for ϕ in $\mathcal{K}$ that justifies ϕ at time point p. The elements of the tuples $(i, L^i_{\text{now}}, F^i_{\text{now}}, F^i_{\text{prev}}, F^i_{\text{next}})$ are selected as follows:

1. L^i_{now} is the largest subset of $BCQ(\phi)$ such that $\mathcal{K}', i \models \bigwedge_{q \in L^i_{\text{now}}} q$,
2. F^i_{now} is the largest subset of $F(\phi)$ such that $\mathcal{K}', i \models F^i_{\text{now}}$,
3. $F^i_{\text{prev}} = F^{i-1}_{\text{now}}$ for $i > 0$,
4. $F^i_{\text{next}} = F^{i+1}_{\text{now}}$ for $i < n$, and
5. $F^0_{\text{prev}} = F^n_{\text{next}} = \emptyset$.

We show that J is correct and justifies ϕ at time point p. The latter case is easy: since $\mathcal{K}', p \models \phi$, we have $\phi \in F^p_{\text{now}}$, and therefore ϕ is justified by J at time point p. It remains to show that J is correct, i.e., that every tuple of J satisfies the nine conditions of the definition of justified tuples.

Conditions 1, 2, 3 and 4 follow straightforwardly from the construction. Condition 5 is satisfied because if $\psi \in F(\phi)$ is such that $\psi \notin F^i_{\text{now}}$, then $\mathcal{K}', i \not\models \psi$ and $F^i_{\text{now}} \not\models \psi$.

For Condition 6, we show that for every $\psi \in F(\phi)$ and for every $0 \leqslant i \leqslant n$, if $\bigwedge_{q \in L^i_{\text{now}}} q \wedge \bigcirc^-(\bigwedge_{\chi \in F^i_{\text{prev}}} \chi) \wedge \bigcirc(\bigwedge_{\chi \in F^i_{\text{next}}} \chi) \models \psi$, then $\mathcal{K}', i \models \psi$, which in turn implies $\psi \in F^i_{\text{now}}$. Since $\mathcal{K}'$ entails every CQ in L^i_{now} at time point i, every TCQ in F^i_{prev} at time point $i-1$, and every TCQ in F^i_{next} at time point $i+1$, every TCQ that corresponds to a formula entailed by L^i_{now}, $\bigcirc^-(\bigwedge_{\chi \in F^i_{\text{prev}}} \chi)$ or $\bigcirc(\bigwedge_{\chi \in F^i_{\text{next}}} \chi)$ is entailed from $\mathcal{K}'$ at time point i. Hence, if $\bigwedge_{q \in L^i_{\text{now}}} q \wedge \bigcirc^-(\bigwedge_{\chi \in F^i_{\text{prev}}} \chi) \wedge \bigcirc(\bigwedge_{\chi \in F^i_{\text{next}}} \chi) \models \psi$, then $\mathcal{K}', i \models \psi$.

For Condition 7, we do a case analysis based on the structure of the elements in F_{now}, using the Proposition 9.12 and the fact that $\mathsf{N_{RC}} = \mathsf{N_{RR}} = \emptyset$.

- If $\mathcal{K}', i \models \psi \vee \psi'$, then $\mathcal{K}', i \models \psi$ or $\mathcal{K}', i \models \psi'$. Therefore, if $\psi \vee \psi' \in F^i_{\text{now}}$, either $\psi \in F^i_{\text{now}}$, or $\psi' \in F^i_{\text{now}}$.
- If $\mathcal{K}', i \models \Diamond\psi$, then $\mathcal{K}', i \models \psi$ or $\mathcal{K}', i+1 \models \Diamond\psi$. Therefore, if $\Diamond\psi \in F^i_{\text{now}}$, either $\psi \in F^i_{\text{now}}$ or $\Diamond\psi \in F^{i+1}_{\text{now}} = F^i_{\text{next}}$.
- If $\mathcal{K}', i \models \Diamond^b\psi$, then $\mathcal{K}', i \models \psi$ or $\mathcal{K}', i+1 \models \Diamond^b\psi$. Therefore, if $\Diamond^b\psi \in F^i_{\text{now}}$, either $\psi \in F^i_{\text{now}}$ or $\Diamond^b\psi \in F^{i+1}_{\text{now}} = F^i_{\text{next}}$.
- If $\mathcal{K}', i \models \Diamond^-\psi$, then $\mathcal{K}', i \models \psi$ or $\mathcal{K}', i-1 \models \Diamond^-\psi$. Therefore, if $\Diamond^-\psi \in F^i_{\text{now}}$, either $\psi \in F^i_{\text{now}}$ or $\Diamond^-\psi \in F^{i-1}_{\text{now}} = F^i_{\text{prev}}$.
- If $\mathcal{K}', i \models \psi \mathsf{U} \psi'$, then $\mathcal{K}', i \models \psi'$, or $\mathcal{K}', i \models \psi$ and $\mathcal{K}', i+1 \models \psi \mathsf{U} \psi'$. Therefore, if $\psi \mathsf{U} \psi' \in F^i_{\text{now}}$, either $\psi' \in F^i_{\text{now}}$, or $\psi \in F^i_{\text{now}}$ and $\psi \mathsf{U} \psi' \in F^i_{\text{next}}$.
- If $\mathcal{K}', i \models \psi \mathsf{U}^b \psi'$, then $\mathcal{K}', i \models \psi'$ or $\mathcal{K}', i \models \psi$ and $\mathcal{K}', i+1 \models \psi \mathsf{U}^b \psi'$. Therefore, if $\psi \mathsf{U}^b \psi' \in F^i_{\text{now}}$, either $\psi' \in F^i_{\text{now}}$, or $\psi \in F^i_{\text{now}}$ and $\psi \mathsf{U}^b \psi' \in F^i_{\text{next}}$.
- If $\mathcal{K}', i \models \psi \mathsf{S} \psi'$, then $\mathcal{K}', i \models \psi'$, or $\mathcal{K}', i \models \psi$ and $\mathcal{K}', i-1 \models \psi \mathsf{S} \psi'$. Therefore, if $\psi \mathsf{S} \psi' \in F^i_{\text{now}}$, then either $\psi' \in F^i_{\text{now}}$, or $\psi \in F^i_{\text{now}}$ and $\psi \mathsf{S} \psi' \in F^i_{\text{prev}}$.
- If ψ is of the form $\Box\varphi$ and $\psi \in F_{\text{now}}$, i.e., $\mathcal{K}', i \models \Box\varphi$, then for every $j > n$, $\mathcal{K}', j \models \varphi$. Since there are no past operators in φ, and no BCQ is entailed from $\mathcal{K}'$ at time point $j > n$ in the absence of rigid predicates, the only possibility is that φ is trivially entailed at any time point $j > n$. It follows that $\langle \emptyset, (\emptyset)_{0 \leqslant i \leqslant n} \rangle, n+1 \models \Box\varphi$.

The proof of Condition 8 is as follows.

- If $\psi \in F(\phi)$ is of the form $\CIRCLE^b \varphi$, then $\mathcal{K}', n \models \psi$ and also $\psi \in F^n_{\text{now}}$.
- Assume $\psi \in F(\phi)$ is of the form $\bigcirc\varphi$ and such that $\langle \emptyset, (\emptyset)_{0 \leqslant i \leqslant n} \rangle, n+1 \not\models \Box\varphi$. Note that, because $\langle \emptyset, (\emptyset)_{0 \leqslant i \leqslant n} \rangle, n+1 \not\models \Box\varphi$, φ cannot be trivially entailed at any time point $j > n$, and consequently also not at $n+1$. In the absence of rigid roles, and because ψ cannot contain past operators, we therefore have $\mathcal{K}', n+1 \not\models \varphi$, which implies $\mathcal{K}', n \not\models \bigcirc\varphi$ and $\psi \notin F^n_{\text{now}}$
- if $\varphi \in F^n_{\text{now}}$, then $\mathcal{K}', n \models \varphi$, which in turn implies $\mathcal{K}', n \models \Diamond\varphi$, $\mathcal{K}', n \models \Diamond^b\varphi$, $\mathcal{K}', n \models \Box^b\varphi$, $\mathcal{K}', n \models \varphi' \mathsf{U} \varphi$ and $\mathcal{K}', n \models \varphi' \mathsf{U}^b \varphi$. It follows that if any of those entailed TCQs are in $F(\phi)$, then they are also in F^n_{now} For the other direction, we do a case analysis.
  * Assume $\Diamond\varphi \in F^n_{\text{now}}$ and $\langle \emptyset, (\emptyset)_{0 \leqslant i \leqslant n} \rangle, n+1 \not\models \Box\varphi$. Note that, because $\langle \emptyset, (\emptyset)_{0 \leqslant i \leqslant n} \rangle, n+1 \not\models \Box\varphi$, φ cannot be trivially entailed at any time point $j > n$. If this would be the case, due to the absence of rigid predicates, φ would also be trivially entailed at all time points $j > n$. Because $\mathcal{K}', n \models \Diamond\varphi$, we must have $\mathcal{K}', n \models \varphi$, since φ cannot be entailed at any time point $j > n$ in the absence of rigid predicates, and ii) φ does not contain past operators. Therefore, $\varphi \in F^n_{\text{now}}$.
  * If $\Diamond^b\varphi \in F^n_{\text{now}}$, then $\mathcal{K}', n \models \Diamond^b\varphi$, which in turn implies $\mathcal{K}', n \models \varphi$ and $\varphi \in F^n_{\text{now}}$.
  * If $\Box^b\varphi \in F^n_{\text{now}}$, then $\mathcal{K}', n \models \Box^b\varphi$, which in turn implies $\mathcal{K}', n \models \varphi$ and $\varphi \in F^n_{\text{now}}$.
  * Assume $\varphi' \mathsf{U} \varphi \in F^n_{\text{now}}$ and $\langle \emptyset, (\emptyset)_{0 \leqslant i \leqslant n} \rangle, n+1 \not\models \Box\varphi$. Similar to the case for $\Diamond$, then $\mathcal{K}', n \models \varphi' \mathsf{U} \varphi$ and $\mathcal{K}', n \models \varphi$, because i) φ cannot be entailed at time point $j > n$ in the absence of rigid predicates, and ii) φ does not contain past operators. Consequently, $\varphi \in F^n_{\text{now}}$.
  * If $\varphi' \mathsf{U}^b \varphi \in F^n_{\text{now}}$, also $\mathcal{K}', n \models \varphi' \mathsf{U}^b \varphi$, which in turn implies $\mathcal{K}', n \models \varphi$ and $\varphi \in F^n_{\text{now}}$.

Condition 9 can be shown similarly to Condition 8.

We have shown that every tuple in J is justified, and consequently that J is correct and justifies ϕ at p. $\square$

A.2. Proofs of Section 9

The following properties of $chase_{\text{rig}}(\mathcal{K})$ will be useful for the proofs of Subsection 9.1.

Proposition A.5. *$chase_{rig}(\mathcal{K})$ satisfies the following properties.*

(P1) $x^{i_1}_{aP_1} \in \Gamma_N \implies P_1(a, x^{i_1}_{aP_1}) \in chase^{\mathcal{K}}_{rig}(\mathcal{A}_{i_1})$

(P2) $x^{i_1 \dots i_l}_{aP_1 \dots P_l} \in \Gamma_N, l > 1 \implies P_l(x^{i_1 \dots i_{l-1}}_{aP_1 \dots P_{l-1}}, x^{i_1 \dots i_l}_{aP_1 \dots P_l}) \in chase^{\mathcal{K}}_{rig}(\mathcal{A}_{i_l})$

(P3) $chase^{\mathcal{K}}_{rig}(\mathcal{A}_i) \models B(x^{i_1 \dots i_l}_{aP_1 \dots P_l}) \implies \mathcal{T} \models \exists P^-_l \sqsubseteq B$

(P4) $x^{i_1 \dots i_l}_{aP_1 \dots P_l} \in \Gamma_N, l > 1 \implies \mathcal{T} \models \exists P^-_{l-1} \sqsubseteq \exists P_l$

(P5) $chase^{\mathcal{K}}_{rig}(\mathcal{A}_i) \models B(a), a \in \mathsf{N}^{\mathcal{K}}_{\mathsf{I}} \implies \langle \mathcal{T}, \mathcal{A}_i \rangle \models B(a)$ *or there exists* $B' := A|\exists R|\exists R^-$ *with* $A \in \mathsf{N}_{\mathsf{RC}}, R \in \mathsf{N}_{\mathsf{RR}}$ *such that* $\mathcal{T} \models B' \sqsubseteq B$ *and there exists* j *such that* $\langle \mathcal{T}, \mathcal{A}_j \rangle \models B'(a)$

(P6) $chase^{\mathcal{K}}_{rig}(\mathcal{A}_i) \models B(x^{i_1 \dots i_l}_{aP_1 \dots P_l}) \implies i = i_l$ *or there exists* $B' := A|\exists R|\exists R^-$ *with* $A \in \mathsf{N}_{\mathsf{RC}}, R \in \mathsf{N}_{\mathsf{RR}}$ *such that* $\mathcal{T} \models B' \sqsubseteq B$ *and* $chase^{\mathcal{K}}_{rig}(\mathcal{A}_{i_l}) \models B'(x^{i_1 \dots i_l}_{aP_1 \dots P_l})$

(P7) $P(a, b) \in chase^{\mathcal{K}}_{rig}(\mathcal{A}_i), a, b \in \mathsf{N}^{\mathcal{K}}_{\mathsf{I}} \implies \langle \mathcal{T}, \mathcal{A}_i \rangle \models P(a, b)$ *or there exists* $P' := R|R^-$ *with* $R \in \mathsf{N}_{\mathsf{RR}}$ *such that* $\mathcal{T} \models P' \sqsubseteq P$ *and there exists* j *such that* $\langle \mathcal{T}, \mathcal{A}_j \rangle \models P'(a, b)$

(P8) $P(a, x^{i_1}_{aP_1}) \in chase^{\mathcal{K}}_{rig}(\mathcal{A}_i), a \in \mathsf{N}^{\mathcal{K}}_{\mathsf{I}}, i_1 = i \implies \mathcal{T} \models P_1 \sqsubseteq P$ *and* $\langle \mathcal{T}, \mathcal{A}_i \rangle \models \exists x. P_1(a, x)$ *or there exists* $B := A|\exists R|\exists R^-$ *with* $A \in \mathsf{N}_{\mathsf{RC}}, R \in \mathsf{N}_{\mathsf{RR}}$ *such that* $\mathcal{T} \models B \sqsubseteq \exists P_1$ *and there exists* j *such that* $\langle \mathcal{T}, \mathcal{A}_j \rangle \models B(a)$

(P9) $P(a, x^{i_1}_{aP_1}) \in chase^{\mathcal{K}}_{rig}(\mathcal{A}_i), a \in \mathsf{N}^{\mathcal{K}}_{\mathsf{I}}, i_1 \neq i \implies$ *there exists* $P' := R|R^-$ *with* $R \in \mathsf{N}_{\mathsf{RR}}$ *such that* $\mathcal{T} \models P_1 \sqsubseteq P' \sqsubseteq P$

(P10) $P(x, y) \in chase^{\mathcal{K}}_{rig}(\mathcal{A}_i), x, y \in \Gamma_N \implies x = x^{i_1 \dots i_l}_{aP_1 \dots P_l}, y = x^{i_1 \dots i_l i_{l+1}}_{aP_1 \dots P_l P_{l+1}}$ *and* $\mathcal{T} \models P_{l+1} \sqsubseteq P$ *or* $x = x^{i_1 \dots i_l i_{l+1}}_{aP_1 \dots P_l P_{l+1}}, y = x^{i_1 \dots i_l}_{aP_1 \dots P_l}$ *and* $\mathcal{T} \models P_{l+1} \sqsubseteq P^-$

(P11) $P(x^{i_1 \dots i_l}_{aP_1 \dots P_l}, x^{i_1 \dots i_l i_{l+1}}_{aP_1 \dots P_l P_{l+1}}) \in chase^{\mathcal{K}}_{rig}(\mathcal{A}_i), i_{l+1} \neq i \implies$ *there exists* $P' := R|R^-$ *with* $R \in \mathsf{N}_{\mathsf{RR}}$ *such that* $\mathcal{T} \models P_{l+1} \sqsubseteq P' \sqsubseteq P$ *and* $P'(x^{i_1 \dots i_l}_{aP_1 \dots P_l}, x^{i_1 \dots i_l i_{l+1}}_{aP_1 \dots P_l P_{l+1}}) \in chase^{\mathcal{K}}_{rig}(\mathcal{A}_{i_{l+1}})$

(P12) $P_l(x^{i_1 \dots i_{l-1}}_{aP_1 \dots P_{l-1}}, x^{i_1 \dots i_l}_{aP_1 \dots P_l}) \in chase^{\mathcal{K}}_{rig}(\mathcal{A}_{i_l}) \implies \exists j, \langle \mathcal{T}, \mathcal{A}_j \rangle \models \exists xy. P_{l-1}(x, y)$

Proof. We refer to [49] for the detailed proof of these properties. $\square$

Lemma 9.3. *If $\mathcal{K}$ is consistent, then $\mathcal{J}_{\mathcal{K}}$ is a model of $\mathcal{K}$ that respects rigid predicates.*

Proof. We first show that $\mathcal{J}_{\mathcal{K}}$ is a model of $\mathcal{K}$, i.e., that for every $i \in [\![0, n]\!]$, $\mathcal{I}_i \models \mathcal{A}_i$ and for every $i \geqslant 0$, $\mathcal{I}_i \models \mathcal{T}$. It is easy to see that for every $i \in [\![0, n]\!]$, $\mathcal{I}_i \models \mathcal{A}_i$ because $\mathcal{A}_i \subseteq chase^{\mathcal{K}}_{\text{rig}}(\mathcal{A}_i)$. We can show that $\mathcal{I}_i$ satisfies every positive inclusion of $\mathcal{T}$ with similar arguments as those used in [48]. We only consider the case $i \leqslant n + 1$ explicitly. For the case $i > n + 1$, we assume $\mathcal{A}_i$ to be replaced by $\mathcal{A}_{n+1}$ in what follows. If a PI $\alpha \in \mathcal{T}_p$ is not satisfied, there is an assertion $\beta \in chase^{\mathcal{K}}_{\text{rig}}(\mathcal{A}_i)$ such that α is applicable to β in $chase^{\mathcal{K}}_{\text{rig}}(\mathcal{A}_i)$. This is impossible given that every PI applicable to β in $\mathcal{S}^j_i$ at step j of the construction of the rigid chase becomes not applicable to β in $\mathcal{S}^k_i$ for some $k \geqslant j$. Indeed, because each PI can only be applied once to a given assertion, there are only finitely many assertions before β, and only finitely many PIs are applied to the assertions that precede β. Finally, we show that because $\mathcal{K}$ is consistent, $\mathcal{I}_i$ satisfies every negative inclusion of $\mathcal{T}$. Indeed, if a negative inclusion would not be satisfied, this would imply the existence of a conflict $\mathcal{B}$ in $chase^{\mathcal{K}}_{\text{rig}}(\mathcal{A}_i)$. If $\mathcal{B} = \{\alpha\}$, the timed assertion $(\alpha', j) \in (\mathcal{A}_i)_{0 \leqslant i \leqslant n}$ from which α has been derived by applying PIs from $\mathcal{T}_p$ is clearly inconsistent. Otherwise $\mathcal{B} = \{\alpha, \beta\}$ with α derived from (α', j) and β derived from (β', k). If $j = k$, $\{(\alpha', j), (\beta', k)\}$ is clearly inconsistent. If $j \neq k$, since α and β belong to $chase^{\mathcal{K}}_{\text{rig}}(\mathcal{A}_i)$, if $j \neq i$ (resp. $k \neq i$), there exists $\alpha'' \in chase^{\mathcal{K}}_{\text{rig}}(\mathcal{A}_i)$ rigid such that α derives from α'', which derives from α' (resp. $\beta'' \in chase^{\mathcal{K}}_{\text{rig}}(\mathcal{A}_i)$ rigid such that β derives from β'', which derives from β'). Therefore, and because no sequence of interpretations that respects rigid predicates can be a model of $\{(\alpha', j), (\beta', k)\}$ and $\mathcal{T}$, $\{(\alpha', j), (\beta', k)\}$ is inconsistent.

Moreover, the model $\mathcal{J}_{\mathcal{K}}$ respects rigid predicates, because if an assertion β of $chase^{\mathcal{K}}_{\text{rig}}(\mathcal{A}_i)$ is rigid, either

$\beta \in \mathcal{A}_i$ and by construction $\beta \in \mathcal{S}^0_k = \mathcal{A}'_k$ for every k, or β has been derived at some step j by applying some PI to an assertion of $\mathcal{S}^j$ and $\beta \in \mathcal{S}^{j+1}_k$ for every k, so that in both cases $\beta \in chase^{\mathcal{K}}_{\text{rig}}(\mathcal{A}_k)$ for every k. □

Lemma 9.4. *If $\mathcal{K}$ is consistent, then for every BTCQ without negation ϕ such that $\mathsf{N}^{\phi}_{\mathsf{I}} \subseteq \mathsf{N}^{\mathcal{K}}_{\mathsf{I}}$, $\mathcal{K}, p \models \phi$ iff $\mathcal{J}_{\mathcal{K}}, p \models \phi$.*

Proof. Since $\mathcal{J}_{\mathcal{K}} = (\mathcal{I}_i)_{i \geqslant 0}$ with $\mathcal{I}_i = \langle \Delta, \cdot^{\mathcal{I}_i} \rangle$ is a model of $\mathcal{K}$ that respects rigid predicates, the first direction is clear, and we only need to show that $\mathcal{J}_{\mathcal{K}}, p \models \phi$ implies $\mathcal{K}, p \models \phi$. Let $\mathcal{J} = (\mathcal{I}'_i)_{i \geqslant 0}$ with $\mathcal{I}'_i = \langle \Delta', \cdot^{\mathcal{I}'_i} \rangle$ be a model of $\mathcal{K}$ that respects rigid predicates. We show by structural induction on ϕ that if $\mathcal{J}_{\mathcal{K}}, p \models \phi$, then $\mathcal{J}, p \models \phi$.

If ϕ is a BCQ $\exists \vec{y}.\psi(\vec{y})$, we show that if there exists a homomorphism π of $\exists \vec{y}.\psi(\vec{y})$ into $\mathcal{I}_p$, then $\mathcal{I}'_p \models \exists \vec{y}.\psi(\vec{y})$. We define a mapping h from Δ into Δ', where we assume w.l.o.g. that Δ and Δ' are disjoint.

1. For every $a \in \mathsf{N}^{\mathcal{K}}_{\mathsf{I}}$, set $h(a^{\mathcal{I}_p}) = a^{\mathcal{I}'_p}$.
2. For every $x^{i_1}_{aP_1} \in \Gamma_N$, set $h(x^{i_1 \mathcal{I}_p}_{aP_1}) = y$, where $(a^{\mathcal{I}'_p}, y) \in P_1^{\mathcal{I}'_{i_1}}$. (If there are several such y, choose one of them randomly.)
3. For every $x^{i_1 \dots i_l}_{aP_1 \dots P_l} \in \Gamma_N$ with $l > 1$, set $h(x^{i_1 \dots i_l \mathcal{I}_p}_{aP_1 \dots P_l}) = y$, where $(h(x^{i_1 \dots i_{l-1} \mathcal{I}_p}_{aP_1 \dots P_{l-1}}), y) \in P_l^{\mathcal{I}'_{i_l}}$. (If there are several such y, choose one of them randomly.)

We first show that h is well defined, i.e., that in the two latter cases, there always exists a y as required. We show this by induction on l. For $l = 1$, because $x^{i_1}_{aP_1} \in \Gamma_N$, by (P1) $P_1(a, x^{i_1}_{aP_1}) \in chase^{\mathcal{K}}_{\text{rig}}(\mathcal{A}_{i_1})$. Therefore, by (P8), either (i) $\langle \mathcal{T}, \mathcal{A}_{i_1} \rangle \models \exists x.P_1(a, x)$, and since $\mathcal{I}'_{i_1}$ is a model of $\langle \mathcal{T}, \mathcal{A}_{i_1} \rangle$, there is some $(a^{\mathcal{I}'_p}, y) \in P_1^{\mathcal{I}'_{i_1}}$, or (ii) there exists $B := A|\exists R|\exists R^-$ with $A \in \mathsf{N}_{\mathsf{RC}}$, $R \in \mathsf{N}_{\mathsf{RR}}$, such that $\mathcal{T} \models B \sqsubseteq \exists P_1$, and there exists j such that $\langle \mathcal{T}, \mathcal{A}_j \rangle \models B(a)$. In the latter case, since $\mathcal{J}$ is a model of $\mathcal{K}$ that respects rigid predicates, $\mathcal{I}'_{i_1} \models B(a)$. Since $\mathcal{I}'_{i_1}$ is a model of $\mathcal{T}$, there is some $(a^{\mathcal{I}'_p}, y) \in P_1^{\mathcal{I}'_{i_1}}$. Then, for $l > 1$, since $x^{i_1 \dots i_l}_{aP_1 \dots P_l} \in \Gamma_N$, by (P4), $\mathcal{T} \models \exists P^-_{l-1} \sqsubseteq \exists P_l$. Since by induction there is an $(x, h(x^{i_1 \dots i_{l-1} \mathcal{I}_p}_{aP_1 \dots P_{l-1}})) \in P_{l-1}^{\mathcal{I}'_{i_l}}$, it follows that there is some $(h(x^{i_1 \dots i_{l-1} \mathcal{I}_p}_{aP_1 \dots P_{l-1}}), y) \in P_l^{\mathcal{I}'_{i_l}}$.

Next, we show that h is a homomorphism of $\mathcal{I}_p$ into $\mathcal{I}'_p$, which then implies that $h \circ \pi$ is a homomorphism of $\exists \vec{y}.\psi(\vec{y})$ into $\mathcal{I}'_p$. We only consider the case $p \leqslant n$ explicitly, and assume $\mathcal{A}_p$ to be replaced with $\mathcal{A}_{n+1}$ for the case $p > n$.

For every $a \in \mathsf{N}^{\mathcal{K}}_{\mathsf{I}}$ and concept A, if $a^{\mathcal{I}_p} \in A^{\mathcal{I}_p}$, i.e., $A(a) \in chase^{\mathcal{K}}_{\text{rig}}(\mathcal{A}_p)$, then by (P5), either (i) $\langle \mathcal{T}, \mathcal{A}_p \rangle \models A(a)$, and since $\mathcal{I}'_p$ is a model of $\langle \mathcal{T}, \mathcal{A}_p \rangle$, also $h(a^{\mathcal{I}_p}) = a^{\mathcal{I}'_p} \in A^{\mathcal{I}'_p}$, or (ii) there exists a concept $B = C|\exists R|\exists R^-$ with $C \in \mathsf{N}_{\mathsf{RC}}$, $R \in \mathsf{N}_{\mathsf{RR}}$, such that $\mathcal{T} \models B \sqsubseteq A$ and there exists j such that $\langle \mathcal{T}, \mathcal{A}_j \rangle \models B(a)$. In the latter case, since $\mathcal{J}$ is a model of $\mathcal{K}$ that respects rigid predicates, $\mathcal{I}'_p \models B(a)$. Since $\mathcal{I}'_p$ is a model of $\mathcal{T}$, it follows that $\mathcal{I}'_p \models A(a)$, so $h(a^{\mathcal{I}_p}) = a^{\mathcal{I}'_p} \in A^{\mathcal{I}'_p}$. For every pair $a, b \in \mathsf{N}^{\mathcal{K}}_{\mathsf{I}}$ and role P, if $(a^{\mathcal{I}_p}, b^{\mathcal{I}_p}) \in P^{\mathcal{I}_p}$, by (P7), similar arguments can be used to prove that $(h(a^{\mathcal{I}_p}), h(b^{\mathcal{I}_p})) = (a^{\mathcal{I}'_p}, b^{\mathcal{I}'_p}) \in P^{\mathcal{I}'_p}$.

For every $x^{i_1 \dots i_l}_{aP_1 \dots P_l} \in \Gamma_N$, such that $x^{i_1 \dots i_l \mathcal{I}_p}_{aP_1 \dots P_l} \in A^{\mathcal{I}_p}$, i.e., $A(x^{i_1 \dots i_l}_{aP_1 \dots P_l}) \in chase^{\mathcal{K}}_{\text{rig}}(\mathcal{A}_p)$, by (P6) we are in one of the following cases.

1. $i_l = p$. By (P3), $\mathcal{T} \models \exists P^-_l \sqsubseteq A$ and by construction of h, $h(x^{i_1 \dots p \mathcal{I}_p}_{aP_1 \dots P_l}) = y$ with $(h(x^{i_1 \dots i_{l-1} \mathcal{I}_p}_{aP_1 \dots P_{l-1}}), y) \in P_l^{\mathcal{I}'_p}$ (note that if $l = 1$, $x^{i_1 \dots i_{l-1}}_{aP_1 \dots P_{l-1}} = a$). Since $\mathcal{I}'_p$ is a model of $\mathcal{T}$, it follows that $y \in A^{\mathcal{I}'_p}$.
2. There exists $B := C|\exists R|\exists R^-$ with $C \in \mathsf{N}_{\mathsf{RC}}, R \in \mathsf{N}_{\mathsf{RR}}$ such that $\mathcal{T} \models B \sqsubseteq A$ and $chase^{\mathcal{K}}_{\text{rig}}(\mathcal{A}_{i_l}) \models B(x^{i_1 \dots i_l}_{aP_1 \dots P_l})$. As in case (i), by (P3) and definition of h we have that $h(x^{i_1 \dots i_l \mathcal{I}_p}_{aP_1 \dots P_l}) = y \in B^{\mathcal{I}'_{i_l}}$. Since B is rigid, $y \in B^{\mathcal{I}'_p}$. Since $\mathcal{I}'_p$ is a model of $\mathcal{T}$, it follows that $y \in A^{\mathcal{I}'_p}$.

For every pair $x, y \in \Gamma_N$ and role P such that $(x^{\mathcal{I}_p}, y^{\mathcal{I}_p}) \in P^{\mathcal{I}_p}$, by (P10) $x = x^{i_1 \dots i_l}_{aP_1 \dots P_l}$, $y = x^{i_1 \dots i_l i_{l+1}}_{aP_1 \dots P_l P_{l+1}}$ and $\mathcal{T} \models P_{l+1} \sqsubseteq P$, or $x = x^{i_1 \dots i_l i_{l+1}}_{aP_1 \dots P_l P_{l+1}}$, $y = x^{i_1 \dots i_l}_{aP_1 \dots P_l}$ and $\mathcal{T} \models P_{l+1} \sqsubseteq P^-$. We can assume w.l.o.g. that we are in the first case. (Otherwise we consider $(y^{\mathcal{I}_p}, x^{\mathcal{I}_p}) \in P^{-\mathcal{I}_p}$.) If $i_{l+1} = p$, by definition of h, $(h(x^{\mathcal{I}_p}), h(y^{\mathcal{I}_p})) \in P_{l+1}^{\mathcal{I}'_p}$, and since $\mathcal{I}'_p$ is a model of $\mathcal{T}$, $(h(x^{\mathcal{I}_p}), h(y^{\mathcal{I}_p})) \in P^{\mathcal{I}'_p}$. Otherwise, by (P11), there exists $P' := R|R^-$ with $R \in \mathsf{N}_{\mathsf{RR}}$ such that $\mathcal{T} \models P_{l+1} \sqsubseteq P' \sqsubseteq P$ and $P'(x, y) \in chase^{\mathcal{K}}_{\text{rig}}(\mathcal{A}_{i_{l+1}})$. With the same arguments as in the first case, we show that $(h(x^{\mathcal{I}_p}), h(y^{\mathcal{I}_p})) \in P'^{\mathcal{I}'_{i_{l+1}}}$, and since P' is rigid $(h(x^{\mathcal{I}_p}), h(y^{\mathcal{I}_p})) \in P'^{\mathcal{I}'_p}$. Since $\mathcal{I}'_p$ is a model of $\mathcal{T}$, it follows that $(h(x^{\mathcal{I}_p}), h(y^{\mathcal{I}_p})) \in P^{\mathcal{I}'_p}$.

Finally, if $a \in \mathsf{N}^{\mathcal{K}}_{\mathsf{I}}$ and $x \in \Gamma_N$, then $(a^{\mathcal{I}_p}, x^{\mathcal{I}_p}) \in P^{\mathcal{I}_p}$ only if $x = x^{i_1}_{aP_1}$. If $i_1 = p$, by definition of h, $(h(a^{\mathcal{I}_p}), h(x^{\mathcal{I}_p})) \in P_1^{\mathcal{I}'_p}$. Since by (P8), $\mathcal{T} \models$

$P_1 \sqsubseteq P$ and $\mathcal{I}'_p$ is a model of $\mathcal{T}$, it follows that $(h(a^{\mathcal{I}_p}), h(x^{\mathcal{I}_p})) \in P^{\mathcal{I}'_p}$. If $i_1 \neq p$, by (P9), there exists P' rigid such that $\mathcal{T} \models P_1 \sqsubseteq P' \sqsubseteq P$, and since by definition of h, $(h(a^{\mathcal{I}_p}), h(x^{\mathcal{I}_p})) \in P_1^{\mathcal{I}'_{i_1}}$, then $(h(a^{\mathcal{I}_p}), h(x^{\mathcal{I}_p})) \in P'^{\mathcal{I}'_{i_1}}$. Since $\mathcal{J}$ respects rigid predicates, it follows that $(h(a^{\mathcal{I}_p}), h(x^{\mathcal{I}_p})) \in P'^{\mathcal{I}'_p}$ and $(h(a^{\mathcal{I}_p}), h(x^{\mathcal{I}_p})) \in P^{\mathcal{I}'_p}$.

We have thus shown that $\mathcal{J}_{\mathcal{K}}, p \models \exists \vec{y}.\psi(\vec{y})$ implies $\mathcal{J}, p \models \exists \vec{y}.\psi(\vec{y})$.

Now for the inductive step, assume that for two BTCQs ϕ_1, ϕ_2 such that $\mathsf{N}_\mathsf{I}^{\phi_1} \subseteq \mathsf{N}_\mathsf{I}^{\mathcal{K}}$ and $\mathsf{N}_\mathsf{I}^{\phi_2} \subseteq \mathsf{N}_\mathsf{I}^{\mathcal{K}}$, we have that $\mathcal{J}_{\mathcal{K}}, p \models \phi_i$ implies $\mathcal{J}, p \models \phi_i$ ($i \in \{1,2\}$). We show that then, for every BTCQ ϕ that we can construct in one step from ϕ_1 and ϕ_2, $\mathcal{J}_{\mathcal{K}}, p \models \phi$ also implies $\mathcal{J}, p \models \phi$. We distinguish the cases based on ϕ.

- If $\mathcal{J}_{\mathcal{K}}, p \models \phi_1 \wedge \phi_2$, then $\mathcal{J}_{\mathcal{K}}, p \models \phi_1$ and $\mathcal{J}_{\mathcal{K}}, p \models \phi_2$, and therefore by assumption, $\mathcal{J}, p \models \phi_1$ and $\mathcal{J}, p \models \phi_2$. Hence, $\mathcal{J}, p \models \phi_1 \wedge \phi_2$.
- If $\mathcal{J}_{\mathcal{K}}, p \models \phi_1 \vee \phi_2$, then $\mathcal{J}_{\mathcal{K}}, p \models \phi_1$ or $\mathcal{J}_{\mathcal{K}}, p \models \phi_2$, and therefore by assumption, $\mathcal{J}, p \models \phi_1$ or $\mathcal{J}, p \models \phi_2$. Hence, $\mathcal{J}, p \models \phi_1 \vee \phi_2$.
- If $\mathcal{J}_{\mathcal{K}}, p \models \bigcirc\phi_1$, then $\mathcal{J}_{\mathcal{K}}, p+1 \models \phi_1$, and therefore by assumption, $\mathcal{J}, p+1 \models \phi_1$. Hence, $\mathcal{J}, p \models \bigcirc\phi_1$.
- In the same way as in the last case, we can show that $\mathcal{J}_{\mathcal{K}}, p \models \bullet^b \phi_1$ implies $\mathcal{J}, p \models \bullet^b \phi_1$, that $\mathcal{J}_{\mathcal{K}}, p \models \bigcirc^- \phi_1$ implies $\mathcal{J}, p \models \bigcirc^- \phi_1$, and that $\mathcal{J}_{\mathcal{K}}, p \models \bullet^- \phi_1$ implies $\mathcal{J}, p \models \bullet^- \phi_1$.
- If $\mathcal{J}_{\mathcal{K}}, p \models \Box\phi_1$, then for every $k \geqslant p$, $\mathcal{J}_{\mathcal{K}}, k \models \phi_1$, and therefore, by assumption, for every $k \geqslant p$, $\mathcal{J}, k \models \phi_1$. Hence, $\mathcal{J}, p \models \Box\phi_1$.
- In the same way as in the last case, we can show that $\mathcal{J}_{\mathcal{K}}, p \models \Box^b\phi_1$ implies $\mathcal{J}, p \models \Box^b\phi_1$ and that $\mathcal{J}_{\mathcal{K}}, p \models \Box^-\phi_1$ implies $\mathcal{J}, p \models \Box^-\phi_1$.
- If $\mathcal{J}_{\mathcal{K}}, p \models \Diamond\phi_1$, then there exists $k \geqslant p$, $\mathcal{J}_{\mathcal{K}}, k \models \phi_1$, and therefore by assumption $\mathcal{J}, k \models \phi_1$. Hence, $\mathcal{J}, p \models \Diamond\phi_1$.
- In the same way as in the last case, we can show that $\mathcal{J}_{\mathcal{K}}, p \models \Diamond^b\phi_1$ implies $\mathcal{J}, p \models \Diamond^b\phi_1$, and that $\mathcal{J}_{\mathcal{K}}, p \models \Diamond^-\phi_1$ implies $\mathcal{J}, p \models \Diamond^-\phi_1$.
- If $\mathcal{J}_{\mathcal{K}}, p \models \phi_1 \mathsf{U}\, \phi_2$, then there exists $k \geqslant p$ such that $\mathcal{J}_{\mathcal{K}}, k \models \phi_2$ and for every j, $p \leqslant j < k$, $\mathcal{J}_{\mathcal{K}}, j \models \phi_1$. Therefore, by assumption $\mathcal{J}, k \models \phi_2$, and for every j, $p \leqslant j < k$, $\mathcal{J}, j \models \phi_1$. Hence, $\mathcal{J}, p \models \phi_1 \mathsf{U}\, \phi_2$.
- We can show in the same way as in the last case that $\mathcal{J}_{\mathcal{K}}, p \models \phi_1 \mathsf{U}^b \phi_2$ implies $\mathcal{J}, p \models \phi_1 \mathsf{U}^b \phi_2$, and that $\mathcal{J}_{\mathcal{K}}, p \models \phi_1 \mathsf{S}\, \phi_2$ implies $\mathcal{J}, p \models \phi_1 \mathsf{S}\, \phi_2$.

We conclude by induction that for every BTCQ ϕ without negation such that $\mathsf{N}_\mathsf{I}^{\phi} \subseteq \mathsf{N}_\mathsf{I}^{\mathcal{K}}$, $\mathcal{J}_{\mathcal{K}}, p \models \phi$ implies $\mathcal{J}, p \models \phi$. It follows that $\mathcal{J}_{\mathcal{K}}, p \models \phi$ implies $\mathcal{K}, p \models \phi$.

We have thus shown that for every BTCQ ϕ without negation such that $\mathsf{N}_\mathsf{I}^{\phi} \subseteq \mathsf{N}_\mathsf{I}^{\mathcal{K}}$, we have $\mathcal{K}, p \models \phi$ iff $\mathcal{J}_{\mathcal{K}}, p \models \phi$. □

Lemma 9.7. *Let $q = \exists \vec{y}.\psi(\vec{y})$ be such that $\mathsf{N}_\mathsf{I}^{q} \subseteq \mathsf{N}_\mathsf{I}^{\mathcal{K}}$. For every $p \in [\![0, n]\!]$, if $\mathcal{K}_{\mathcal{R}}, p \models q$ then $\mathcal{K}, p \models q$.*

Proof. Assume that $\mathcal{K}_{\mathcal{R}}, p \models \exists \vec{y}.\psi(\vec{y})$. By Proposition 3.6, since $\mathsf{N}_{\mathsf{RC}} = \mathsf{N}_{\mathsf{RR}} = \emptyset$, $\langle \mathcal{T}, (\mathcal{A}_p \cup \mathcal{R})\rangle \models \exists \vec{y}.\psi(\vec{y})$. Let $\mathcal{I}_p^{\mathcal{R}} = \langle \Delta^{\mathcal{I}_p^{\mathcal{R}}}, \cdot^{\mathcal{I}_p^{\mathcal{R}}} \rangle$ be the canonical model of $\langle \mathcal{T}, (\mathcal{A}_p \cup \mathcal{R})\rangle$. There exists a homomorphism π of $\exists \vec{y}.\psi(\vec{y})$ into $\mathcal{I}_p^{\mathcal{R}}$. We first define a mapping σ from $\{x^{\mathcal{I}_p^{\mathcal{R}}} \mid x \in \mathsf{N}_\mathsf{I}^{\mathcal{K}}$ or occurs in $\mathcal{R}\}$ into $\{x^{\mathcal{I}_p} \mid x \in \mathsf{N}_\mathsf{I}^{\mathcal{K}} \cup \Gamma_N, x$ occurs in $\mathit{chase}_{\mathrm{rig}}^{\mathcal{K}}(\mathcal{A}_p)\}$, where we assume Δ and $\Delta^{\mathcal{I}_p^{\mathcal{R}}}$ to be disjoint, by

- $\sigma(a^{\mathcal{I}_p^{\mathcal{R}}}) = a^{\mathcal{I}_p}$ for $a \in \mathsf{N}_\mathsf{I}^{\mathcal{K}}$,
- $\sigma(x_{aP}^{\mathcal{I}_p^{\mathcal{R}}}) = x^{\mathcal{I}_p}$ such that $P(a,x) \in \mathit{chase}_{\mathrm{rig}}^{\mathcal{K}}(\mathcal{A}_p)$,
- $\sigma(x_{P}^{\mathcal{I}_p^{\mathcal{R}}}) = x^{\mathcal{I}_p}$ such that there exists $P(y,x) \in \bigcup_{i=0}^{n} \mathit{chase}_{\mathrm{rig}}^{\mathcal{K}}(\mathcal{A}_i)$, and
- $\sigma(x_{PP'}^{\mathcal{I}_p^{\mathcal{R}}}) = x^{\mathcal{I}_p}$ such that $P'(y,x) \in \mathit{chase}_{\mathrm{rig}}^{\mathcal{K}}(\mathcal{A}_p)$ with $\sigma(x_{P}^{\mathcal{I}_p^{\mathcal{R}}}) = y^{\mathcal{I}_p}$.

Claim 1. σ is well defined.

Proof of claim. If x_{aP} occurs in $\mathcal{R}$, there exists i such that $\langle \mathcal{T}, \mathcal{A}_i\rangle \models \exists x.P(a,x)$. Since $\mathcal{I}_i$ is a model of $\langle \mathcal{T}, \mathcal{A}_i\rangle$, it follows that there is some $P(a,x) \in \mathit{chase}_{\mathrm{rig}}^{\mathcal{K}}(\mathcal{A}_i)$. Moreover, since P is rigid, $P(a,x) \in \mathit{chase}_{\mathrm{rig}}^{\mathcal{K}}(\mathcal{A}_p)$.

If x_P occurs in $\mathcal{R}$, there exists i such that $\langle \mathcal{T}, \mathcal{A}_i\rangle \models \exists xy.P(x,y)$. Since $\mathcal{I}_i$ is a model of $\langle \mathcal{T}, \mathcal{A}_i\rangle$, it follows that there exist $x, y \in \mathsf{N}_\mathsf{I}^{\mathcal{K}} \cup \Gamma_N$ such that $P(y,x) \in \mathit{chase}_{\mathrm{rig}}^{\mathcal{K}}(\mathcal{A}_i)$. Moreover, x occurs in $\mathit{chase}_{\mathrm{rig}}^{\mathcal{K}}(\mathcal{A}_p)$ because there exists $B := A|\exists R|\exists R^-$ with $A \in \mathsf{N}_{\mathsf{RC}}$ and $R \in \mathsf{N}_{\mathsf{RR}}$ such that $\mathcal{T} \models \exists P^- \sqsubseteq B$, and therefore there is a rigid assertion $\beta \models B(x)$ such that $\beta \in \mathit{chase}_{\mathrm{rig}}^{\mathcal{K}}(\mathcal{A}_p)$.

If $x_{PP'}$ occurs in $\mathcal{R}$, then x_P also occurs in $\mathcal{R}$. It follows that there exist i and $y \in \mathsf{N}_\mathsf{I}^{\mathcal{K}} \cup \Gamma_N$ such that $P(y, \sigma(x_{P}^{\mathcal{I}_p^{\mathcal{R}}})) \in \mathit{chase}_{\mathrm{rig}}^{\mathcal{K}}(\mathcal{A}_i)$. Moreover, by construction of $\mathcal{R}$, P' is rigid and such that $\mathcal{T} \models \exists P^- \sqsubseteq \exists P'$. Since $\mathcal{I}_i$ is a model of $\mathcal{T}$, there then exists $x \in \mathsf{N}_\mathsf{I}^{\mathcal{K}} \cup \Gamma_N$ such that $P'(\sigma(x_{P}^{\mathcal{I}_p^{\mathcal{R}}}), x) \in \mathit{chase}_{\mathrm{rig}}^{\mathcal{K}}(\mathcal{A}_i)$. Hence, $P'(\sigma(x_{P}^{\mathcal{I}_p^{\mathcal{R}}}), x) \in \mathit{chase}_{\mathrm{rig}}^{\mathcal{K}}(\mathcal{A}_p)$. ■

Claim 2. σ is a partial homomorphism of $\mathcal{I}_p^{\mathcal{R}}$ into $\mathcal{I}_p$.

Proof of claim. For every $a \in \mathsf{N}_\mathsf{I}^\mathcal{K}$ and concept A, if $a^{\mathcal{I}_p^\mathcal{R}} \in A^{\mathcal{I}_p^\mathcal{R}}$, since $\mathcal{I}_p^\mathcal{R}$ is the canonical model of $\langle \mathcal{T}, (\mathcal{A}_p \cup \mathcal{R})\rangle$, then $\langle \mathcal{T}, (\mathcal{A}_p \cup \mathcal{R})\rangle \models A(a)$. Let $\{\alpha\}$ be a cause for $A(a)$. If $\alpha \in \mathcal{A}_p$, then $\alpha \in chase_{\text{rig}}^\mathcal{K}(\mathcal{A}_p)$. In this case, since $\mathcal{I}_p$ is a model of $\mathcal{T}$ and $\langle \mathcal{T}, \alpha\rangle \models A(a)$, then $\sigma(a^{\mathcal{I}_p^\mathcal{R}}) = a^{\mathcal{I}_p} \in A^{\mathcal{I}_p}$. Otherwise, $\alpha \in \mathcal{R}$, and α is either of the form $A'(a)$ with $A' \in \mathsf{N}_\mathsf{RC}$, or of the form $P(a,b)$ or $P(a, x_{aP})$, where P is rigid. In the first two cases, there exists i such that $\langle \mathcal{T}, \mathcal{A}_i\rangle \models \alpha$. Therefore, since $\mathcal{I}_i$ is a model of $\langle \mathcal{T}, \mathcal{A}_i\rangle$, $\alpha \in chase_{\text{rig}}^\mathcal{K}(\mathcal{A}_i)$. Since α is rigid, $\alpha \in chase_{\text{rig}}^\mathcal{K}(\mathcal{A}_p)$, and therefore, since $\mathcal{I}_p$ is a model of $\mathcal{T}$ and $\langle \mathcal{T}, \alpha\rangle \models A(a)$, we obtain that $\sigma(a^{\mathcal{I}_p^\mathcal{R}}) = a^{\mathcal{I}_p} \in A^{\mathcal{I}_p}$. In the last case, if $\alpha = P(a, x_{aP})$, there exists i such that $\langle \mathcal{T}, \mathcal{A}_i\rangle \models \exists x.P(a,x)$. Since $\mathcal{I}_i$ is a model of $\langle \mathcal{T}, \mathcal{A}_i\rangle$, there is some $P(a,x) \in chase_{\text{rig}}^\mathcal{K}(\mathcal{A}_i)$. Since P is rigid, $P(a,x) \in chase_{\text{rig}}^\mathcal{K}(\mathcal{A}_p)$, and since $\mathcal{I}_p$ is a model of $\mathcal{T}$ and $\langle \mathcal{T}, P(a,x)\rangle \models A(a)$, we obtain $\sigma(a^{\mathcal{I}_p^\mathcal{R}}) = a^{\mathcal{I}_p} \in A^{\mathcal{I}_p}$.

For every pair $a, b \in \mathsf{N}_\mathsf{I}^\mathcal{K}$ and role P, if $(a^{\mathcal{I}_p^\mathcal{R}}, b^{\mathcal{I}_p^\mathcal{R}}) \in P^{\mathcal{I}_p^\mathcal{R}}$, we can use similar arguments to show that $(\sigma(a^{\mathcal{I}_p^\mathcal{R}}), \sigma(b^{\mathcal{I}_p^\mathcal{R}})) = (a^{\mathcal{I}_p}, b^{\mathcal{I}_p}) \in P^{\mathcal{I}_p}$.

For every x_{aP} that occurs in $\mathcal{R}$ and $A \in \mathsf{N}_\mathsf{C}$, if $x_{aP}^{\mathcal{I}_p^\mathcal{R}} \in A^{\mathcal{I}_p^\mathcal{R}}$, since $\mathcal{I}_p^\mathcal{R}$ is the canonical model of $\langle \mathcal{T}, (\mathcal{A}_p \cup \mathcal{R})\rangle$, then $\langle \mathcal{T}, (\mathcal{A}_p \cup \mathcal{R})\rangle \models A(x_{aP})$. Let $\{\alpha\}$ be a cause for $A(x_{aP})$. By construction, the only assertion of $\mathcal{A}_p \cup \mathcal{R}$ that involves x_{aP} is $P(a, x_{aP})$. Therefore, $\alpha = P(a, x_{aP})$ and $\langle \mathcal{T}, P(a, x_{aP})\rangle \models A(x_{aP})$. Since $\sigma(x_{aP}^{\mathcal{I}_p^\mathcal{R}}) = x^{\mathcal{I}_p}$ is such that $P(a,x) \in chase_{\text{rig}}^\mathcal{K}(\mathcal{A}_p)$, and $\mathcal{I}_p$ is a model of $\mathcal{T}$, then $\sigma(x_{aP}^{\mathcal{I}_p^\mathcal{R}}) \in A^{\mathcal{I}_p}$.

For every $a \in \mathsf{N}_\mathsf{I}^\mathcal{K}$, $x \notin \mathsf{N}_\mathsf{I}^\mathcal{K}$ that occurs in $\mathcal{R}$ and role P, if $(a^{\mathcal{I}_p^\mathcal{R}}, x^{\mathcal{I}_p^\mathcal{R}}) \in P^{\mathcal{I}_p^\mathcal{R}}$, since $\mathcal{I}_p^\mathcal{R}$ is the canonical model of $\langle \mathcal{T}, (\mathcal{A}_p \cup \mathcal{R})\rangle$, then $\langle \mathcal{T}, (\mathcal{A}_p \cup \mathcal{R})\rangle \models P(a,x)$. Let $\{\alpha\}$ be a cause for $P(a,x)$. By construction of $\mathcal{R}$, $x = x_{aP_1}$ and $\alpha = P_1(a, x_{aP_1})$, and by definition of σ, $(\sigma(a^{\mathcal{I}_p^\mathcal{R}}), \sigma(x_{aP_1P}^{\mathcal{I}_p^\mathcal{R}})) \in P_1^{\mathcal{I}_p}$. Since $\langle \mathcal{T}, P_1(a,x)\rangle \models P(a,x)$ and $\mathcal{I}_p$ is a model of $\mathcal{T}$, it follows that $(\sigma(a^{\mathcal{I}_p^\mathcal{R}}), \sigma(x_{aP_1P}^{\mathcal{I}_p^\mathcal{R}})) \in P^{\mathcal{I}_p}$.

For every x_{P_1} that occurs in $\mathcal{R}$ and $A \in \mathsf{N}_\mathsf{C}$, if $x_{P_1}^{\mathcal{I}_p^\mathcal{R}} \in A^{\mathcal{I}_p^\mathcal{R}}$, since $\mathcal{I}_p^\mathcal{R}$ is the canonical model of $\langle \mathcal{T}, (\mathcal{A}_p \cup \mathcal{R})\rangle$, then $\langle \mathcal{T}, (\mathcal{A}_p \cup \mathcal{R})\rangle \models A(x_{P_1})$. Let $\{\alpha\}$ be a cause for $A(x_{P_1})$. By construction, either $\alpha = A'(x_{P_1})$ with $A' \in \mathsf{N}_\mathsf{RC}$ and $\mathcal{T} \models \exists P_1^- \sqsubseteq A'$, or $\alpha = P_2(x_{P_1}, x_{P_1P_2})$ with P_2 rigid and $\mathcal{T} \models \exists P_1^- \sqsubseteq \exists P_2$. Since $\sigma(x_{P_1}^{\mathcal{I}_p^\mathcal{R}}) = x^{\mathcal{I}_p}$ is such that there exists i such that $P_1(y,x) \in chase_{\text{rig}}^\mathcal{K}(\mathcal{A}_i)$ and $\mathcal{I}_i$ is a model of $\mathcal{T}$, then $A'(x) \in chase_{\text{rig}}^\mathcal{K}(\mathcal{A}_i)$ (resp. there is some $P_2(x,z) \in chase_{\text{rig}}^\mathcal{K}(\mathcal{A}_i)$). Therefore $A'(x) \in chase_{\text{rig}}^\mathcal{K}(\mathcal{A}_p)$ (resp. there is some $P_2(x,z) \in chase_{\text{rig}}^\mathcal{K}(\mathcal{A}_p)$). Because $\mathcal{I}_p$ is a model of $\mathcal{T}$, it follows that $\sigma(x_{P_1}^{\mathcal{I}_p^\mathcal{R}}) \in A^{\mathcal{I}_p}$.

For every $x_{P_1P_2}$ that occurs in $\mathcal{R}$ and $A \in \mathsf{N}_\mathsf{C}$, if $x_{P_1P_2}^{\mathcal{I}_p^\mathcal{R}} \in A^{\mathcal{I}_p^\mathcal{R}}$, since $\mathcal{I}_p^\mathcal{R}$ is the canonical model of $\langle \mathcal{T}, (\mathcal{A}_p \cup \mathcal{R})\rangle$, then $\langle \mathcal{T}, (\mathcal{A}_p \cup \mathcal{R})\rangle \models A(x_{P_1P_2})$. Let $\{\alpha\}$ be a cause for $A(x_{P_1P_2})$. By construction, $\alpha = P_2(x_{P_1}, x_{P_1P_2})$, P_2 is rigid, and $\mathcal{T} \models \exists P_1^- \sqsubseteq \exists P_2$. Since $\sigma(x_{P_1P_2}^{\mathcal{I}_p^\mathcal{R}}) = x^{\mathcal{I}_p}$ is such that there exists $P_2(y,x) \in chase_{\text{rig}}^\mathcal{K}(\mathcal{A}_p)$ (with $y^{\mathcal{I}_p^\mathcal{R}} = \sigma(x_{P_1}^{\mathcal{I}_p^\mathcal{R}})$) and $\mathcal{I}_p$ is a model of $\mathcal{T}$, then $\sigma(x_{P_1P_2}^{\mathcal{I}_p^\mathcal{R}}) \in A^{\mathcal{I}_p}$.

Finally, for every $x, y \notin \mathsf{N}_\mathsf{I}^\mathcal{K}$ that occur in $\mathcal{R}$ and for every role P, if $(x^{\mathcal{I}_p^\mathcal{R}}, y^{\mathcal{I}_p^\mathcal{R}}) \in P^{\mathcal{I}_p^\mathcal{R}}$, since $\mathcal{I}_p^\mathcal{R}$ is the canonical model of $\langle \mathcal{T}, (\mathcal{A}_p \cup \mathcal{R})\rangle$, then $\langle \mathcal{T}, (\mathcal{A}_p \cup \mathcal{R})\rangle \models P(x,y)$. Let $\{\alpha\}$ be a cause for $P(x,y)$. By construction, $x = x_{P_1}$, $y = x_{P_1P_2}$, $\alpha = P_2(x_{P_1}, x_{P_1P_2})$, and P_2 is rigid, and therefore, as previously, $(\sigma(x_{P_1}^{\mathcal{I}_p^\mathcal{R}}), \sigma(x_{P_1P_2}^{\mathcal{I}_p^\mathcal{R}})) \in P^{\mathcal{I}_p}$. ∎

Claim 3. σ can be extended to a homomorphism σ' of $\mathcal{I}_p^\mathcal{R}$ into $\mathcal{I}_p$.

Proof of claim. Since $\mathcal{I}_p^\mathcal{R}$ is the canonical model of $\langle \mathcal{T}, (\mathcal{A}_p \cup \mathcal{R})\rangle$, $\mathcal{I}_p$ is a model of $\mathcal{T}$, and σ preserves the concept or role memberships, we can extend σ to a homomorphism σ' of $\mathcal{I}_p^\mathcal{R}$ into $\mathcal{I}_p$ by mapping the anonymous part of $\mathcal{I}_p^\mathcal{R}$ rooted in $x^{\mathcal{I}_p^\mathcal{R}} \in \{x^{\mathcal{I}_p^\mathcal{R}} \mid x \in \mathsf{N}_\mathsf{I}^\mathcal{K}$ or occurs in $\mathcal{R}\}$ to the part of $\mathcal{I}_p$ rooted in $\sigma(x^{\mathcal{I}_p^\mathcal{R}})$. ∎

From Claim 3, it follows that $\sigma' \circ \pi$ is a homomorphism of $\exists \vec{y}.\psi(\vec{y})$ into $\mathcal{I}_p$. We have thus shown that if $\mathcal{K}_\mathcal{R}, p \models \exists \vec{y}.\psi(\vec{y})$, then $\mathcal{I}_p \models \exists \vec{y}.\psi(\vec{y})$, i.e., $\mathcal{J}_\mathcal{K}, p \models \exists \vec{y}.\psi(\vec{y})$. Hence, if $\mathcal{K}_\mathcal{R}, p \models \exists \vec{y}.\psi(\vec{y})$, then $\mathcal{K}, p \models \exists \vec{y}.\psi(\vec{y})$. □

Lemma 9.8. *Let $q = \exists \vec{y}.\psi(\vec{y})$ be such that $\mathsf{N}_\mathsf{I}^q \subseteq \mathsf{N}_\mathsf{I}^\mathcal{K}$. For every $p \in [\![0, n]\!]$, if $\mathcal{K}, p \models q$ then $\mathcal{K}_\mathcal{R}, p \models q$.*

Proof. Assume that $\mathcal{K}, p \models \exists \vec{y}.\psi(\vec{y})$. Then, $\mathcal{I}_p \models \exists \vec{y}.\psi(\vec{y})$, and there exists a homomorphism π of $\exists \vec{y}.\psi(\vec{y})$ into $\mathcal{I}_p$. Let $\mathcal{I}_p^\mathcal{R} = \langle \Delta^{\mathcal{I}_p^\mathcal{R}}, \cdot^{\mathcal{I}_p^\mathcal{R}}\rangle$ be a model of $\langle \mathcal{T}, (\mathcal{A}_i \cup \mathcal{R})\rangle$. We define a mapping $h_p^\mathcal{R}$ from $\{x^{\mathcal{I}_p} \mid x \in \mathsf{N}_\mathsf{I}^\mathcal{K} \cup \Gamma_N, x$ occurs in $chase_{\text{rig}}^\mathcal{K}(\mathcal{A}_p)\}$ into $\Delta^{\mathcal{I}_p^\mathcal{R}}$, where we again assume that Δ and $\Delta^{\mathcal{I}_p^\mathcal{R}}$ are disjoint.

– For every $a \in \mathsf{N}_\mathsf{I}^\mathcal{K}$, we set $h_p^\mathcal{R}(a^{\mathcal{I}_p}) = a^{\mathcal{I}_p^\mathcal{R}}$

- For every $x^{i_1}_{aP_1}$, where $i_1 \neq p$ and P_1 is rigid, we set $h^{\mathcal{R}}_p(x^{i_1\mathcal{I}_p}_{aP_1}) = x^{\mathcal{I}^{\mathcal{R}}_p}_{aP_1}$
- For every $x^{i_1\dots i_l}_{aP_1\dots P_l}$ with $l > 1$, such that for every $j \in [\![1,l]\!]$, $i_j \neq p$, P_l is rigid, and P_{l-1} is not rigid, we set $h^{\mathcal{R}}_p(x^{i_1\dots i_l\mathcal{I}_p}_{aP_1\dots P_l}) = x^{\mathcal{I}^{\mathcal{R}}_p}_{P_{l-1}P_l}$
- For every $x^{i_1\dots i_l}_{aP_1\dots P_l}$ with $l > 1$, such that for every $j \in [\![1,l]\!]$, $i_j \neq p$, and P_l and P_{l-1} are rigid, we set $h^{\mathcal{R}}_p(x^{i_1\dots i_l\mathcal{I}_p}_{aP_1\dots P_l}) = y$, where $(h^{\mathcal{R}}_p(x^{i_1\dots i_{l-1}\mathcal{I}_p}_{aP_1\dots P_{l-1}}), y) \in P_l^{\mathcal{I}^{\mathcal{R}}_p}$ If there are several such y, we choose one of them randomly.
- For every $x^{i_1\dots i_l}_{aP_1\dots P_l}$ such that for every $k \in [\![1,l]\!]$, $i_j \neq p$, and P_l not rigid, we set $h^{\mathcal{R}}_p(x^{i_1\dots i_l\mathcal{I}_p}_{aP_1\dots P_l}) = x^{\mathcal{I}^{\mathcal{R}}_p}_{P_l}$
- For every $x^{i_1\dots i_l}_{aP_1\dots P_l}$ such that for some $j \in [\![1,l]\!]$, $i_j = p$, we set $h^{\mathcal{R}}_p(x^{i_1\dots i_l\mathcal{I}_p}_{aP_1\dots P_l}) = y$, where $(h^{\mathcal{R}}_p(x^{i_1\dots i_{l-1}\mathcal{I}_p}_{aP_1\dots P_{l-1}}), y) \in P_l^{\mathcal{I}^{\mathcal{R}}_p}$. If there are several such y, we choose one of them randomly.

Claim 1. $h^{\mathcal{R}}_p$ is well defined.

Proof of claim. We distinguish the cases based on the argument of $h^{\mathcal{R}}_p$.

- Case $x^{i_1}_{aP_1}$ with $i_1 \neq p$ and P_1 is rigid, $h^{\mathcal{R}}_p(x^{i_1\mathcal{I}_p}_{aP_1}) = x^{\mathcal{I}^{\mathcal{R}}_p}_{aP_1}$.
  Since $x^{i_1}_{aP_1} \in \Gamma_N$, by (P1) and (P8), $\exists x.P_1(a,x)$ is entailed by some $\langle \mathcal{T}, \mathcal{A}_j\rangle$. Therefore, x_{aP_1} appears in $\mathcal{R}$.
- Case $x^{i_1\dots i_l}_{aP_1\dots P_l}$ with $l > 1$, such that for every $j \in [\![1,l]\!]$, $i_j \neq p$, P_l is rigid and P_{l-1} is not rigid, $h^{\mathcal{R}}_p(x^{i_1\dots i_l\mathcal{I}_p}_{aP_1\dots P_l}) = x^{\mathcal{I}^{\mathcal{R}}_p}_{P_{l-1}P_l}$.
  Since $x^{i_1\dots i_l}_{aP_1\dots P_l} \in \Gamma_N$, by (P4), $\mathcal{T} \models \exists P^-_{l-1} \sqsubseteq \exists P_l$, and by (P2) and (P12), there is some j such that $\langle \mathcal{T}, \mathcal{A}_j\rangle \models \exists xy.P_{l-1}(x,y)$. Moreover, P_l is rigid and P_{l-1} is not rigid, and therefore $x_{P_{l-1}P_l}$ appears in $\mathcal{R}$.
- Case $x^{i_1\dots i_l}_{aP_1\dots P_l}$ with $l > 1$, such that every $j \in [\![1,l]\!]$, $i_j \neq p$, and P_l and P_{l-1} are rigid, $h^{\mathcal{R}}_p(x^{i_1\dots i_l\mathcal{I}_p}_{aP_1\dots P_l}) = y$, where $(h^{\mathcal{R}}_p(x^{i_1\dots i_{l-1}\mathcal{I}_p}_{aP_1\dots P_{l-1}}), y) \in P_l^{\mathcal{I}^{\mathcal{R}}_p}$.
  We show by induction on the length $length = l - r$ of the sequence of rigid roles $P_r\dots P_{l-1}$ that there is always such a $(h^{\mathcal{R}}_p(x^{i_1\dots i_{l-1}\mathcal{I}_p}_{aP_1\dots P_{l-1}}), y) \in P_l^{\mathcal{I}^{\mathcal{R}}_p}$.
  – If $length = 1$, we are in one of the following cases.
  (i) $r > 1$ and $h^{\mathcal{R}}_p(x^{i_1\dots i_{l-1}\mathcal{I}_p}_{aP_1\dots P_{l-1}}) = x^{\mathcal{I}^{\mathcal{R}}_p}_{P_{l-2}P_{l-1}}$. Then $(x^{\mathcal{I}^{\mathcal{R}}_p}_{P_{l-2}}, x^{\mathcal{I}^{\mathcal{R}}_p}_{P_{l-2}P_{l-1}}) \in P^{\mathcal{I}^{\mathcal{R}}_p}_{l-1}$, because $\mathcal{I}^{\mathcal{R}}_p$ is a model of $\mathcal{R}$. Since $x^{i_1\dots i_l}_{aP_1\dots P_l} \in \Gamma_N$, by (P4), $\mathcal{T} \models \exists P^-_{l-1} \sqsubseteq \exists P_l$. Therefore, since $\mathcal{I}^{\mathcal{R}}_p$ is a model of $\mathcal{T}$, there is some $(h^{\mathcal{R}}_p(x^{i_1\dots i_{l-1}\mathcal{I}_p}_{aP_1\dots P_{l-1}}), y) \in P_l^{\mathcal{I}^{\mathcal{R}}_p}$.
  (ii) $r = 1$ and $h^{\mathcal{R}}_p(x^{i_1\dots i_{l-1}\mathcal{I}_p}_{aP_1\dots P_{l-1}}) = h^{\mathcal{R}}_p(x^{i_1\mathcal{I}_p}_{aP_1}) = x^{\mathcal{I}^{\mathcal{R}}_p}_{aP_1}$ is such that $(a^{\mathcal{I}^{\mathcal{R}}_p}, x^{\mathcal{I}^{\mathcal{R}}_p}_{aP_1}) \in P_1^{\mathcal{I}^{\mathcal{R}}_p}$ because $P_1(a, x_{aP_1}) \in \mathcal{R}$. Since $x^{i_1i_2}_{aP_1P_2} \in \Gamma_N$, $\mathcal{T} \models \exists P^-_1 \sqsubseteq \exists P_2$ by (P4). Therefore, since $\mathcal{I}^{\mathcal{R}}_p$ is a model of $\mathcal{T}$, there is some $(x^{\mathcal{I}^{\mathcal{R}}_p}_{aP_1}, y) \in P_2^{\mathcal{I}^{\mathcal{R}}_p}$.
  – For $length > 1$, $\mathcal{T} \models \exists P^-_{l-1} \sqsubseteq \exists P_l$ by (P4). It follows that, since by inductive hypothesis there is some $(x, h^{\mathcal{R}}_p(x^{i_1\dots i_{l-1}\mathcal{I}_p}_{aP_1\dots P_{l-1}})) \in P^{\mathcal{I}^{\mathcal{R}}_p}_{l-1}$, there then is some $(h^{\mathcal{R}}_p(x^{i_1\dots i_{l-1}\mathcal{I}_p}_{aP_1\dots P_{l-1}}), y) \in P_l^{\mathcal{I}^{\mathcal{R}}_p}$.
- Case $x^{i_1\dots i_l}_{aP_1\dots P_l}$ such that for every $j \in [\![1,l]\!]$, $i_j \neq p$, and P_l not rigid, $h^{\mathcal{R}}_p(x^{i_1\dots i_l\mathcal{I}_p}_{aP_1\dots P_l}) = x^{\mathcal{I}^{\mathcal{R}}_p}_{P_l}$.
  Since $\mathcal{T}$ does not contain any role inclusion of the form $P' \sqsubseteq P$ with $P' := R_1|R^-_1$, $R_1 \in \mathsf{N_R}\backslash\mathsf{N_{RR}}$ and $P := R_2|R^-_2$, $R_2 \in \mathsf{N_{RR}}$, and P_l is not rigid, there is no P such that $P_l \sqsubseteq P$ and P is rigid. Therefore, since $i_l \neq p$, there is no P such that $P(x^{i_1\dots i_{l-1}}_{aP_1\dots P_{l-1}}, x^{i_1\dots i_l}_{aP_1\dots P_l}) \in \mathit{chase}^{\mathcal{K}}_{\mathrm{rig}}(\mathcal{A}_p)$. We obtain that $x^{i_1\dots i_l}_{aP_1\dots P_l}$ occurs in $\mathit{chase}^{\mathcal{K}}_{\mathrm{rig}}(\mathcal{A}_p)$ only if there is $B := A|\exists R|\exists R^-$ with $A \in \mathsf{N_{RC}}$, $R \in \mathsf{N_{RR}}$ such that $\mathit{chase}^{\mathcal{K}}_{\mathrm{rig}}(\mathcal{A}_p) \models B(x^{i_1\dots i_l}_{aP_1\dots P_l})$. By (P3), $\mathcal{T} \models \exists P^-_l \sqsubseteq B$, and by (P2) and (P12), there is some j such that $\langle \mathcal{T}, \mathcal{A}_j\rangle \models \exists xy.P_{l-1}(x,y)$. It follows that x_{P_l} appears in $\mathcal{R}$.
- Case $x^{i_1\dots i_l}_{aP_1\dots P_l}$ such that for some $j \in [\![1,l]\!]$, $i_j = p$, $h^{\mathcal{R}}_p(x^{i_1\dots i_l\mathcal{I}_p}_{aP_1\dots P_l}) = y$, where $(h^{\mathcal{R}}_p(x^{i_1\dots i_{l-1}\mathcal{I}_p}_{aP_1\dots P_{l-1}}), y) \in P_l^{\mathcal{I}^{\mathcal{R}}_p}$.
  We show by induction on the length $length = l - r$ of the chain of roles that links $x^{i_1\dots i_l}_{aP_1\dots P_l}$ to the first individual $x^{i_1\dots i_r}_{aP_1\dots P_r}$ such that $i_r = p$ that there is always such $(h^{\mathcal{R}}_p(x^{i_1\dots i_{l-1}\mathcal{I}_p}_{aP_1\dots P_{l-1}}), y) \in P_l^{\mathcal{I}^{\mathcal{R}}_p}$.
  – If $length = 0$, then $i_l = p$ and there is no $j < l$ such that $i_j = p$. We are thus in one of the following cases. Either (i) $h^{\mathcal{R}}_p(x^{i_1\dots i_{l-1}\mathcal{I}_p}_{aP_1\dots P_{l-1}}) = a^{\mathcal{I}^{\mathcal{R}}_p}$, (ii) $h^{\mathcal{R}}_p(x^{i_1\dots i_{l-1}\mathcal{I}_p}_{aP_1\dots P_{l-1}}) = x^{\mathcal{I}^{\mathcal{R}}_p}_{aP_1}$, (iii) $h^{\mathcal{R}}_p(x^{i_1\dots i_{l-1}\mathcal{I}_p}_{aP_1\dots P_{l-1}}) = x^{\mathcal{I}^{\mathcal{R}}_p}_{P_{l-2}P_{l-1}}$, (iv) $h^{\mathcal{R}}_p(x^{i_1\dots i_{l-1}\mathcal{I}_p}_{aP_1\dots P_{l-1}})$ is such that $(h^{\mathcal{R}}_p(x^{i_1\dots i_{l-2}\mathcal{I}_p}_{aP_1\dots P_{l-2}}), h^{\mathcal{R}}_p(x^{i_1\dots i_{l-1}\mathcal{I}_p}_{aP_1\dots P_{l-1}})) \in P^{\mathcal{I}^{\mathcal{R}}_p}_{l-1}$, or (v) $h^{\mathcal{R}}_p(x^{i_1\dots i_{l-1}\mathcal{I}_p}_{aP_1\dots P_{l-1}}) = x^{\mathcal{I}^{\mathcal{R}}_p}_{P_{l-1}}$.
  (i) If $h^{\mathcal{R}}_p(x^{i_1\dots i_{l-1}\mathcal{I}_p}_{aP_1\dots P_{l-1}}) = a^{\mathcal{I}^{\mathcal{R}}_p}$, by definition of $h^{\mathcal{R}}_p$, $x^{i_1\dots i_{l-1}}_{aP_1\dots P_{l-1}} = a$, and therefore $x^{i_1\dots p}_{aP_1\dots P_l} = x^p_{aP_1}$. Since $x^p_{aP_1} \in \Gamma_N$, by (P1), $P_1(a, x^p_{aP_1}) \in \mathit{chase}^{\mathcal{K}}_{\mathrm{rig}}(\mathcal{A}_p)$. By (P8), either (a) $\langle \mathcal{T}, \mathcal{A}_p\rangle \models \exists x.P_1(a,x)$, and there is some $(a^{\mathcal{I}^{\mathcal{R}}_p}, y) \in P_1^{\mathcal{I}^{\mathcal{R}}_p}$ because $\mathcal{I}^{\mathcal{R}}_p$ is a model of $\langle \mathcal{T}, \mathcal{A}_p\rangle$, or (b) there exists $B := A|\exists R|\exists R^-$ with $A \in \mathsf{N_{RC}}$, $R \in \mathsf{N_{RR}}$, such that $\mathcal{T} \models B \sqsubseteq \exists P_1$ and

there exists j such that $\langle\mathcal{T},\mathcal{A}_j\rangle \models B(a)$. In the latter case, $\mathcal{R} \models B(a)$ by construction of $\mathcal{R}$, and since $\mathcal{I}_p^{\mathcal{R}}$ is a model of $\mathcal{R}$, we obtain $\mathcal{I}_p^{\mathcal{R}} \models B(a)$. Since $\mathcal{I}_p^{\mathcal{R}}$ is a model of $\mathcal{T}$, there is some $(a^{\mathcal{I}_p^{\mathcal{R}}}, y) \in P_1^{\mathcal{I}_p^{\mathcal{R}}}$.

(ii) If $h_p^{\mathcal{R}}(x_{aP_1\dots P_{l-1}}^{i_1\dots i_{l-1}\mathcal{I}_p}) = x_{aP_1}^{\mathcal{I}_p^{\mathcal{R}}}$, by definition of $h_p^{\mathcal{R}}$, $x_{aP_1\dots P_{l-1}}^{i_1\dots i_{l-1}} = x_{aP_1}^{i_1}$ and P_1 is rigid. By (P1), $P_1(a, x_{aP_1}^{i_1}) \in \mathit{chase}_{\text{rig}}^{\mathcal{K}}(\mathcal{A}_{i_1})$, and therefore by (P8), either (a) $\langle\mathcal{T},\mathcal{A}_{i_1}\rangle \models \exists x.P_1(a,x)$ and $P_1(a,x_{aP_1}) \in \mathcal{R}$ since P_1 is rigid, or (b) there exists $B := A|\exists R|\exists R^-$ with $A \in \mathsf{N_{RC}}$, $R \in \mathsf{N_{RR}}$, such that $\mathcal{T} \models B \sqsubseteq \exists P_1$, and there exists j such that $\langle\mathcal{T},\mathcal{A}_j\rangle \models B(a)$. In the latter case, $\langle\mathcal{T},\mathcal{A}_j\rangle \models \exists x.P_1(a,x)$, and therefore $P_1(a,x_{aP_1}) \in \mathcal{R}$. In both cases, $(a^{\mathcal{I}_p^{\mathcal{R}}}, x_{aP_1}^{\mathcal{I}_p^{\mathcal{R}}}) \in P_1^{\mathcal{I}_p^{\mathcal{R}}}$ since $\mathcal{I}_p^{\mathcal{R}}$ is a model of $\mathcal{R}$. Moreover, since $x_{aP_1\dots P_l}^{i_1\dots i_l} = x_{aP_1P_2}^{i_1 p} \in \Gamma_N$, by (P4), $\mathcal{T} \models \exists P_1^- \sqsubseteq \exists P_2$. Therefore, since $\mathcal{I}_p^{\mathcal{R}}$ is a model of $\mathcal{T}$, there is some $(x_{aP_1}^{\mathcal{I}_p^{\mathcal{R}}}, y) \in P_2^{\mathcal{I}_p^{\mathcal{R}}}$.

(iii) If $h_p^{\mathcal{R}}(x_{aP_1\dots P_{l-1}}^{i_1\dots i_{l-1}\mathcal{I}_p}) = x_{P_{l-2}P_{l-1}}^{\mathcal{I}_p^{\mathcal{R}}}$, by definition of $\mathcal{R}$, since $x_{P_{l-2}P_{l-1}}$ appears in $\mathcal{R}$, $P_{l-1}(x_{P_{l-2}}, x_{P_{l-2}P_{l-1}}) \in \mathcal{R}$, and therefore $(x_{P_{l-2}}^{\mathcal{I}_p^{\mathcal{R}}}, x_{P_{l-2}P_{l-1}}^{\mathcal{I}_p^{\mathcal{R}}}) \in P_{l-1}^{\mathcal{I}_p^{\mathcal{R}}}$. Since $x_{aP_1\dots P_l}^{i_1\dots i_l} \in \Gamma_N$, by (P4), $\mathcal{T} \models \exists P_{l-1}^- \sqsubseteq \exists P_l$, and there is some $(x_{aP_{l-2}P_{l-1}}^{\mathcal{I}_p^{\mathcal{R}}}, y) \in P_l^{\mathcal{I}_p^{\mathcal{R}}}$.

(iv) If $(h_p^{\mathcal{R}}(x_{aP_1\dots P_{l-2}}^{i_1\dots i_{l-2}\mathcal{I}_p}), h_p^{\mathcal{R}}(x_{aP_1\dots P_{l-1}}^{i_1\dots i_{l-1}\mathcal{I}_p})) \in P_{l-1}^{\mathcal{I}_p^{\mathcal{R}}}$, since $x_{aP_1\dots P_l}^{i_1\dots i_l} \in \Gamma_N$, by (P4), $\mathcal{T} \models \exists P_{l-1}^- \sqsubseteq \exists P_l$, and there is some $(h_p^{\mathcal{R}}(x_{aP_1\dots P_{l-1}}^{i_1\dots i_{l-1}\mathcal{I}_p}), y) \in P_l^{\mathcal{I}_p^{\mathcal{R}}}$.

(v) If $h_p^{\mathcal{R}}(x_{aP_1\dots P_{l-1}}^{i_1\dots i_{l-1}\mathcal{I}_p}) = x_{P_{l-1}}^{\mathcal{I}_p^{\mathcal{R}}}$, by (P2) and since $i_l = p$, we obtain $P_l(x_{aP_1\dots P_{l-1}}^{i_1\dots i_{l-1}}, x_{aP_1\dots P_l}^{i_1\dots p}) \in \mathit{chase}_{\text{rig}}^{\mathcal{K}}(\mathcal{A}_p)$. By (P6), since $\mathit{chase}_{\text{rig}}^{\mathcal{K}}(\mathcal{A}_p) \models \exists P_l(x_{aP_1\dots P_{l-1}}^{i_1\dots i_{l-1}})$ and $i_{l-1} \neq p$, there exists $B := A|\exists R|\exists R^-$ with $A \in \mathsf{N_{RC}}, R \in \mathsf{N_{RR}}$, such that $\mathcal{T} \models B \sqsubseteq \exists P_l$ and $\mathit{chase}_{\text{rig}}^{\mathcal{K}}(\mathcal{A}_{i_{l-1}}) \models B(x_{aP_1\dots P_{l-1}}^{i_1\dots i_{l-1}})$. By (P3), $\mathcal{T} \models \exists P_{l-1}^- \sqsubseteq B$, and $\mathcal{R} \models B(x_{P_{l-1}})$ (since $x_{P_{l-1}}$ occurs in $\mathcal{R}$ and B is rigid). We obtain that $\langle\mathcal{T},\mathcal{R}\rangle \models \exists x.P_l(x_{P_{l-1}}, x)$. Since $\mathcal{I}_p^{\mathcal{R}}$ is a model of $\langle\mathcal{T},\mathcal{R}\rangle$, there is some $(x_{P_{l-1}}^{\mathcal{I}_p^{\mathcal{R}}}, y) \in P_l^{\mathcal{I}_p^{\mathcal{R}}}$.

– For *length* > 0, since $x_{aP_1\dots P_l}^{i_1\dots i_l} \in \Gamma_N$, by (P4), $\mathcal{T} \models \exists P_{l-1}^- \sqsubseteq \exists P_l$. Since by inductive hypothesis there is some $(x, h_p^{\mathcal{R}}(x_{aP_1\dots P_{l-1}}^{i_1\dots i_{l-1}\mathcal{I}_p})) \in P_{l-1}^{\mathcal{I}_p^{\mathcal{R}}}$, there then is some $(h_p^{\mathcal{R}}(x_{aP_1\dots P_{l-1}}^{i_1\dots i_{l-1}\mathcal{I}_p}), y) \in P_l^{\mathcal{I}_p^{\mathcal{R}}}$. ■

Claim 2. $h_p^{\mathcal{R}}$ is a homomorphism of $\mathcal{I}_p$ into $\mathcal{I}_p^{\mathcal{R}}$.

Proof of claim For every $a \in \mathsf{N_I^{\mathcal{K}}}$ and concept A, if $a^{\mathcal{I}_p} \in A^{\mathcal{I}_p}$, i.e., $A(a) \in \mathit{chase}_{\text{rig}}^{\mathcal{K}}(\mathcal{A}_p)$, then by (P5), either (i) $\langle\mathcal{T},\mathcal{A}_p\rangle \models A(a)$, and since $\mathcal{I}_p^{\mathcal{R}}$ is a model of $\langle\mathcal{T},\mathcal{A}_p\rangle$, then $h_p^{\mathcal{R}}(a^{\mathcal{I}_p}) = a^{\mathcal{I}_p^{\mathcal{R}}} \in A^{\mathcal{I}_p^{\mathcal{R}}}$, or (ii) there exists $B := C|\exists R|\exists R^-$ with $C \in \mathsf{N_{RC}}$, $R \in \mathsf{N_{RR}}$, such that $\mathcal{T} \models B \sqsubseteq A$, and there exists j such that $\langle\mathcal{T},\mathcal{A}_j\rangle \models B(a)$. In the latter case $\mathcal{R} \models B(a)$. Therefore, since $\mathcal{I}_p^{\mathcal{R}}$ is a model of $\mathcal{R}$, $\mathcal{I}_p^{\mathcal{R}} \models B(a)$, and $\mathcal{I}_p^{\mathcal{R}} \models A(a)$ because $\mathcal{I}_p^{\mathcal{R}}$ is a model of $\mathcal{T}$. It follows that $h_p^{\mathcal{R}}(a^{\mathcal{I}_p}) = a^{\mathcal{I}_p^{\mathcal{R}}} \in A^{\mathcal{I}_p^{\mathcal{R}}}$. For every pair $a, b \in \mathsf{N_I^{\mathcal{K}}}$ and role P, if $(a^{\mathcal{I}_p}, b^{\mathcal{I}_p}) \in P^{\mathcal{I}_p}$, by (P7), similar arguments can be used to prove that $(h_p^{\mathcal{R}}(a^{\mathcal{I}_p}), h_p^{\mathcal{R}}(b^{\mathcal{I}_p})) = (a^{\mathcal{I}_p^{\mathcal{R}}}, b^{\mathcal{I}_p^{\mathcal{R}}}) \in P^{\mathcal{I}_p^{\mathcal{R}}}$.

For every $x_{aP_1\dots P_l}^{i_1\dots i_l} \in \Gamma_N$, such that $x_{aP_1\dots P_l}^{i_1\dots i_l\mathcal{I}_p} \in A^{\mathcal{I}_p}$, by (P3), $\mathcal{T} \models \exists P_l^- \sqsubseteq A$, and by construction of $h_p^{\mathcal{R}}$, $h_p^{\mathcal{R}}(x_{aP_1\dots P_l}^{i_1\dots i_l\mathcal{I}_p}) = y$ is such that either (i) there exists $(x,y) \in P_l^{\mathcal{I}_p^{\mathcal{R}}}$, and since $\mathcal{I}_p^{\mathcal{R}}$ is a model of $\mathcal{T}$, we have $y \in A^{\mathcal{I}_p^{\mathcal{R}}}$, or (ii) $y = x_{P_l}^{\mathcal{I}_p^{\mathcal{R}}}$, P_l is not rigid, and for every $j \in [\![1,l]\!]$, $i_j \neq p$. In the latter case, by (P6), there exists $B := C|\exists R|\exists R^-$ with $C \in \mathsf{N_{RC}}$, $R \in \mathsf{N_{RR}}$, such that $\mathcal{T} \models B \sqsubseteq A$ and $\mathit{chase}_{\text{rig}}^{\mathcal{K}}(\mathcal{A}_{i_l}) \models B(x_{aP_1\dots P_l}^{i_1\dots i_l})$. By (P3), $\mathcal{T} \models \exists P_l^- \sqsubseteq B$. Therefore, by construction of $\mathcal{R}$, $\mathcal{R} \models B(x_{P_l})$ and $\langle\mathcal{T},\mathcal{R}\rangle \models A(x_{P_l})$. It follows that $y \in A^{\mathcal{I}_p^{\mathcal{R}}}$.

For every pair $x, y \in \Gamma_N$ and role P, such that $(x^{\mathcal{I}_p}, y^{\mathcal{I}_p}) \in P^{\mathcal{I}_p}$, by (P10), either (i) $x = x_{aP_1\dots P_l}^{i_1\dots i_l}$, $y = x_{aP_1\dots P_lP_{l+1}}^{i_1\dots i_li_{l+1}}$ and $\mathcal{T} \models P_{l+1} \sqsubseteq P$, or (ii) $x = x_{aP_1\dots P_lP_{l+1}}^{i_1\dots i_li_{l+1}}$, $y = x_{aP_1\dots P_l}^{i_1\dots i_l}$ and $\mathcal{T} \models P_{l+1} \sqsubseteq P^-$. We can assume w.l.o.g. that we are in the first case (otherwise, we consider $(y^{\mathcal{I}_p}, x^{\mathcal{I}_p}) \in P^{-\mathcal{I}_p}$). If $i_{l+1} = p$, by definition of $h_p^{\mathcal{R}}$, we have $(h_p^{\mathcal{R}}(x^{\mathcal{I}_p}), h_p^{\mathcal{R}}(y^{\mathcal{I}_p})) \in P_{l+1}^{\mathcal{I}_p^{\mathcal{R}}}$. Otherwise, by (P11), there exists $P' := R|R^-$ with $R \in \mathsf{N_{RR}}$ such that $\mathcal{T} \models P_{l+1} \sqsubseteq P' \sqsubseteq P$ and $P'(x,y) \in \mathit{chase}_{\text{rig}}^{\mathcal{K}}(\mathcal{A}_{i_{l+1}})$. In this case, there are two possibilities.

(i) If P_l is not rigid, given that $\mathcal{T} \models P_{l+1} \sqsubseteq P'$ and P' is rigid, P_{l+1} is rigid by our hypothesis on the TBox. It follows that $h_p^{\mathcal{R}}(y^{\mathcal{I}_p}) = x_{P_lP_{l+1}}^{\mathcal{I}_p^{\mathcal{R}}}$. If there is no $i_j = p$, then $h_p^{\mathcal{R}}(x^{\mathcal{I}_p}) = x_{P_l}^{\mathcal{I}_p^{\mathcal{R}}}$. Therefore, since $P_{l+1}(x_{P_1}, x_{P_lP_{l+1}}) \in \mathcal{R}$, then $(h_p^{\mathcal{R}}(x^{\mathcal{I}_p}), h_p^{\mathcal{R}}(y^{\mathcal{I}_p})) \in P_{l+1}^{\mathcal{I}_p^{\mathcal{R}}}$. Otherwise, there exists $i_j = p$, and we obtain $(h_p^{\mathcal{R}}(x^{\mathcal{I}_p}), h_p^{\mathcal{R}}(y^{\mathcal{I}_p})) \in P_{l+1}^{\mathcal{I}_p^{\mathcal{R}}}$ by definition of $h_p^{\mathcal{R}}$.

(ii) If P_l is rigid, then $h_p^{\mathcal{R}}(y^{\mathcal{I}_p})$ is such that $(h_p^{\mathcal{R}}(x^{\mathcal{I}_p^{\mathcal{R}}}), h_p^{\mathcal{R}}(y^{\mathcal{I}_p^{\mathcal{R}}})) \in P_{l+1}^{\mathcal{I}_p^{\mathcal{R}}}$.

Since in any case, $(h_p^{\mathcal{R}}(x^{\mathcal{I}_p}), h_p^{\mathcal{R}}(y^{\mathcal{I}_p})) \in P_{l+1}^{\mathcal{I}_p^{\mathcal{R}}}$ and $\mathcal{I}_p^{\mathcal{R}}$ is a model of $\mathcal{T}$, we obtain $(h_p^{\mathcal{R}}(x^{\mathcal{I}_p}), h_p^{\mathcal{R}}(y^{\mathcal{I}_p})) \in P^{\mathcal{I}_p^{\mathcal{R}}}$.

Finally, if $a \in \mathsf{N}_\mathsf{I}^{\mathcal{K}}$ and $x \in \Gamma_N$, $(a^{\mathcal{I}_p}, x^{\mathcal{I}_p}) \in P^{\mathcal{I}_p}$ only if $x = x_{aP_1}^{i_1}$. If $i_1 = p$, by definition of $h_p^{\mathcal{R}}$, $(h_p^{\mathcal{R}}(a^{\mathcal{I}_p}), h_p^{\mathcal{R}}(x^{\mathcal{I}_p})) \in P_1^{\mathcal{I}_p^{\mathcal{R}}}$, and since by (P8), $\mathcal{T} \models P_1 \sqsubseteq P$, we obtain $(h_p^{\mathcal{R}}(a^{\mathcal{I}_p}), h_p^{\mathcal{R}}(x^{\mathcal{I}_p})) \in P^{\mathcal{I}_p^{\mathcal{R}}}$. If $i_1 \neq p$, by (P9), there exists P' rigid such that $\mathcal{T} \models P_1 \sqsubseteq P' \sqsubseteq P$, so by our hypothesis on the TBox, P_1 is rigid. By (P1) and (P8), there is some j such that $\langle \mathcal{T}, \mathcal{A}_j \rangle \models \exists x.P_1(a, x)$. Therefore, $P_1(a, x_{aP_1}) \in \mathcal{R}$ and $(h_p^{\mathcal{R}}(a^{\mathcal{I}_p}), h_p^{\mathcal{R}}(x^{\mathcal{I}_p})) = (a^{\mathcal{I}_p^{\mathcal{R}}}, x_{aP_1}^{\mathcal{I}_p^{\mathcal{R}}}) \in P_1^{\mathcal{I}_p^{\mathcal{R}}}$. We obtain $(h_p^{\mathcal{R}}(a^{\mathcal{I}_p}), h_p^{\mathcal{R}}(x^{\mathcal{I}_p})) \in P^{\mathcal{I}_p^{\mathcal{R}}}$. ■

It follows from Claim 2 that $h_p^{\mathcal{R}} \circ \pi$ is a homomorphism of $\exists \vec{y}.\psi(\vec{y})$ into $\mathcal{I}_p^{\mathcal{R}}$. Therefore, we have shown that if $\mathcal{I}_p \models \exists \vec{y}.\psi(\vec{y})$, then $\mathcal{K}_{\mathcal{R}}, p \models \exists \vec{y}.\psi(\vec{y})$. This means that if $\mathcal{K}, p \models \exists \vec{y}.\psi(\vec{y})$, then $\mathcal{K}_{\mathcal{R}}, p \models \exists \vec{y}.\psi(\vec{y})$. □